Online Supplement to "On Time-varying Panel Data Models with Time-varying Interactive Fixed Effects"

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Abstract

This online supplement is composed of three sections. Sections A and B provide the proofs of the main results in Sections 3 and 4, respectively. Section C provides the proof of the technical lemmas in Appendix A.

A Proofs of the Main Results in Section 3

This appendix provides the proofs of the main results in Section 3. We need some technical lemmas whose proofs are available in Appendix C.

Recall that $\hat{V}_{NT}^{(r)}$ denotes the diagonal matrix of the first R largest eigenvalues of $(NT)^{-1}\sum_{i=1}^{N}(Y_i^{(r)}-X_i^{(r)}\hat{\beta}_r)(Y_i^{(r)}-X_i^{(r)}\hat{\beta}_r)'$ arranged in decreasing order along its main diagonal line and $H^{(r)}=(N^{-1}\Lambda_r'\Lambda_r)(T^{-1}F^{(r)'}\hat{F}^{(r)})\hat{V}_{NT}^{(r)-1}$. Let V_{NT} denote the diagonal matrix of the first R largest eigenvalues of $(NT)^{-1}\sum_{i=1}^{N}(Y_i-X_i\tilde{\beta})(Y_i-X_i\tilde{\beta})'$ arranged in decreasing order along its main diagonal line and $H=(N^{-1}\Lambda'\Lambda)(T^{-1}F'\tilde{F})V_{NT}^{-1}$. Let $C_{NT}=\min\{\sqrt{Th},\sqrt{N},\,h^{-2}\}$ and $C_{0NT}=\min\{\sqrt{T},\sqrt{N}\}$. Recall that we use $\mathbf{X}_p^{(r)}$ to denote the pth sheet of the $T\times N\times P$ matrix $\mathbf{X}^{(r)}$ and $X_i^{(r)}=(X_{i1}^{(r)},...,X_{iT}^{(r)})'$. With a little bit of abuse of notation, we also write $X_t^{(r)}=(X_{1t}^{(r)},...,X_{Nt}^{(r)})'$. Similarly, let $\mathbf{X}_p^{(r)}$, $X_i^{(r)}$, and $X_t^{(r)}$ denote the non-kernel weighted version of $\mathbf{X}_p$, X_i, and X_t which are respectively, $T\times N$, $T\times P$, and $N\times P$ matrices. Let $\Delta_{it}^{(r)}=k_{h,tr}^{*1/2}\Delta_i(t,r)$ and $\Delta_i^{(r)}=(\Delta_{i1}^{(r)},...,\Delta_{iT}^{(r)})'$. Define

$$\begin{aligned} B_{1t}^{(r)} &= k_{h,tr}^{*1/2}C_{1t}^{(r)}\frac{t-r}{T}+k_{h,tr}^{*1/2}C_{2t}^{(r)}(\frac{t-r}{T})^2,\ B_{2t,1}^{(r)}=N^{-1}\hat{V}_{NT}^{(r)-1}H^{(r)}E(F_tA_{2,tr}')\Lambda_rF_tk_{h,tr}^{*1/2}h^2\kappa_2,\\ B_{2t,2}^{(r)} &= \hat{V}_{NT}^{(r)-1}\frac{1}{TN}\sum_s(\frac{s-r}{T})^2k_{h,sr}^{*}E[\bar{C}_{1s}^{(r)}A_{1,sr}']\Lambda_rF_t^{(r)},\ \text{and}\ B_{2t}^{(r)}=B_{2t,1}^{(r)}+B_{2t,2}^{(r)}, \end{aligned} \tag{A.1}$$

where $C_{1t}^{(r)}$, $C_{2t}^{(r)}$, $\bar{C}_{1t}^{(r)}$, $A_{1,tr}$ and $A_{2,tr}$ are defined before Theorem 3.2 in the paper. Let $\hat{\delta}_r=\hat{\beta}_r-\beta_r$. Let $A_{l,itr}$ denote the ith element of $A_{l,tr}=X_t\beta_r^{(l)}+\Lambda_r^{(l)}F_t$ for $l=1,2$.

Lemma A.1 *Let $\mathcal{F}=\{F\in\mathbb{R}^{T\times R}:F'F/T=\mathbb{I}_R\}$. Let $F^{(r)0}$ and λ_{ir}^0 denote the true values of $F^{(r)}$ and λ_{ir}. Suppose that Assumptions A.1–A.4 hold. Then*

(i) $\sup_{F^{(r)}\in\mathcal{F}}\left\|\frac{1}{NT}\sum_{i=1}^{N}X_i^{(r)\prime}M_{F^{(r)}}\varepsilon_i^{(r)}\right\|=O_P(T^{-1/2}+N^{-1/4})$,

(ii) $\sup_{F^{(r)}\in\mathcal{F}}\left\|\frac{1}{NT}\sum_{i=1}^{N}\lambda_{ir}^{0\prime}F^{(r)0\prime}M_{F^{(r)}}\varepsilon_i\right\| = O_P(T^{-1/2}+N^{-1/4})$,

(iii) $\sup_{F^{(r)}\in\mathcal{F}}\left\|\frac{1}{NT}\sum_{i=1}^{N}\varepsilon_i^{(r)\prime}(P_{F^{(r)}}-P_{F^{(r)0}})\varepsilon_i^{(r)}\right\| = O_P(T^{-1/2}+N^{-1/4})$,

(iv) $\sup_{F^{(r)}\in\mathcal{F}}\left\|\frac{1}{NT}\sum_{i=1}^{N}\Delta_i^{(r)\prime}(P_{F^{(r)}}-P_{F^{(r)0}})\Delta_i^{(r)}\right\| = O_P(h^2)$,

(v) $\sup_{F^{(r)}\in\mathcal{F}}\left\|\frac{1}{NT}\sum_{i=1}^{N}X_i^{(r)\prime}M_{F^{(r)}}\Delta_i^{(r)}\right\| = O_P(h)$,

(vi) $\sup_{F^{(r)}\in\mathcal{F}}\left\|\frac{1}{NT}\sum_{i=1}^{N}\Delta_i^{(r)}M_{F^{(r)}}F^{(r)0}\lambda_{ir}\right\| = O_P(h)$,

(vii) $\sup_{F^{(r)}\in\mathcal{F}}\left\|\frac{1}{NT}\sum_{i=1}^{N}\Delta_i^{(r)\prime}(P_{F^{(r)}}-P_{F^{(r)0}})\varepsilon_i^{(r)}\right\| = O_P((T^{-1/2}+N^{-1/4})h)$.

Lemma A.2 *Suppose that Assumptions A.1–A.4 hold. Then*

(i) $T^{-1}\hat{F}^{(r)\prime}[(NT)^{-1}\sum_{i=1}^{N}(Y_i^{(r)}-X_i^{(r)}\hat{\beta}_r)(Y_i^{(r)}-X_i^{(r)}\hat{\beta}_r)']\hat{F}^{(r)} = \hat{V}_{NT}^{(r)} = V_r + O_P(C_{NT}^{-1}) + O_P(\|\hat{\delta}_r\|)$,

(ii) $T^{-1}\hat{F}^{(r)\prime}F^{(r)} = Q_r + O_P(C_{NT}^{-1}) + O_P(\|\hat{\delta}_{rr}\|)$,

(iii) $H^{(r)} = Q_r^{-1} + O_P(C_{NT}^{-1}) + O_P(\|\hat{\delta}_r\|)$,

(iv) $H^{(r)}H^{(r)\prime} = \Sigma_F^{-1} + O_P(C_{NT}^{-1}) + O_P(\|\hat{\delta}_r\|)$,

(v) $\left\|P_{\hat{F}^{(r)}}-P_{F^{(r)}}\right\|^2 = O_P(C_{NT}^{-2}) + O_P(\|\hat{\delta}_r\|)$,

where V_r is the diagonal matrix consisting of the eigenvalues of $\Sigma_{\Lambda_r}^{1/2}\Sigma_F\Sigma_{\Lambda_r}^{1/2}$ in descending order with Υ_r being the corresponding (normalized) eigenvector matrix, and $Q_r = V_r^{1/2}\Upsilon_r^{-1}\Sigma_{\Lambda_r}^{-1/2}$.

Lemma A.3 *Suppose that Assumptions A.1–A.4 hold. Then*

(i) $T^{-1}\left\|\hat{F}^{(r)}-F^{(r)}H^{(r)}-B^{(r)}\right\|^2 = O_P(C_{NT}^{-2}) + O_P(\|\hat{\delta}_r\|)$.

(ii) $T^{-1}\left\|(\hat{F}^{(r)}-F^{(r)}H^{(r)}-B^{(r)})'F^{(r)}H^{(r)}\right\| = O_P(C_{NT}^{-2}) + O_P(\|\hat{\delta}_r\|)$.

(iii) $T^{-1}\left\|(\hat{F}^{(r)}-F^{(r)}H^{(r)}-B^{(r)})'\hat{F}^{(r)}\right\| = O_P(C_{NT}^{-2}) + O_P(\|\hat{\delta}_r\|)$.

(iv) $T^{-1}\left\|(\hat{F}^{(r)}-F^{(r)}H^{(r)}-B^{(r)})'X_i\right\| = O_P(C_{NT}^{-2}) + O_P(\|\hat{\delta}_r\|)$ *for* $i=1,2,\cdots,N$.

(v) $T^{-1}\left\|(\hat{F}^{(r)}-F^{(r)}H^{(r)}-B^{(r)})'\Delta_i^{(r)}\right\| = O_P(C_{NT}^{-2}h^2) + O_P(\|\hat{\delta}_r\|h^2)$ *for* $i=1,2,\cdots,N$.

Lemma A.4 *Let* $\psi_{NT}^{(r)} = \frac{1}{N}\sum_{i=1}^{N}\sum_{k=1}^{N}\frac{1}{T}X_i^{(r)\prime}F^{(r)}[\frac{1}{T}F^{(r)\prime}F^{(r)}]^{-1}(\frac{1}{N}\Lambda_r'\Lambda_r)^{-1}\lambda_{kr}(\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}^{(r)}\varepsilon_{kt}^{(r)})$. *Suppose that Assumptions A.1–A.4 hold. Then*

(i) $N^{-1}T^{-1}\left\|\sum_{i=1}^{N}X_i^{(r)\prime}M_{\hat{F}^{(r)}}[\hat{F}^{(r)}-F^{(r)}H^{(r)}-B^{(r)}]\lambda_{ir}'\right\| = O_P(\|\hat{\delta}_r\|) + O_P(C_{NT}^{-2})$,

(ii) $\frac{1}{NT}\sum_{i=1}^{N}X_i^{(r)\prime}M_{\hat{F}^{(r)}}B^{(r)}\lambda_{ir} = O_P(h^2) + O_P(\|\hat{\delta}_r\|)$,

(iii) $N^{-1}T^{-1}\left\|\sum_{j=1}^{N}\lambda_{jr}\varepsilon_j^{(r)\prime}\hat{F}^{(r)}\right\| = O_P(\|\hat{\delta}_r\|) + O_P(C_{NT}^{-2})$,

(iv) $\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}[\hat{F}^{(r)}-F^{(r)}H^{(r)}-B^{(r)}H^{(r)}]'\varepsilon_i^{(r)}\right\|^2 = O_P(C_{NT}^{-4}+\|\hat{\delta}_r\|^2)$,

(v) $\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\frac{1}{T}X_i^{(r)\prime}F^{(r)}[\frac{1}{T}F^{(r)\prime}F^{(r)}]^{-1}[\hat{F}^{(r)}H^{(r)-1}-F^{(r)}-B^{(r)}H^{(r)-1}]'\varepsilon_i^{(r)} = \sqrt{\frac{Th}{N}}\psi_{NT}^{(r)} + O_P(\|\hat{\delta}_r\|)$ $+\sqrt{Th}O_P(C_{NT}^{-2})$.

Lemma A.5 *Suppose that Assumptions A.1–A.4 hold. Then*

$$\begin{aligned}&\frac{1}{NT}\sum_{i=1}^{N}X_i^{(r)\prime}M_{\hat{F}^{(r)}}F^{(r)}\lambda_{ir}\\ =\;&\frac{1}{N^2T}\sum_{i=1}^{N}\sum_{j=1}^{N}X_i^{(r)\prime}M_{\hat{F}^{(r)}}X_j^{(r)}a_{ji}^{(r)}\hat{\delta}_r-\frac{1}{N^2T}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}^{(r)}X_i^{(r)\prime}M_{\hat{F}^{(r)}}\varepsilon_j^{(r)}+\frac{1}{Th}\zeta_{NT}^{(r)}\\ &-\frac{1}{N^2T}\sum_{i=1}^{N}\sum_{j=1}^{N}X_i^{(r)\prime}M_{\hat{F}^{(r)}}\Delta_j^{(r)}a_{ji}^{(r)}+O_P(C_{NT}^{-1}+\|\hat{\delta}_r\|^{1/2})O_P(\|\hat{\delta}_r\|)+O_P(C_{NT}^{-3}+h^4+C_{NT}^{-1}h^3),\end{aligned}$$

where $\zeta_{NT}^{(r)} = -\frac{h}{N^2T}\sum_{i=1}^{N}\sum_{j=1}^{N}X_i^{(r)\prime}M_{\hat{F}^{(r)}}\varepsilon_j^{(r)}\varepsilon_j^{(r)\prime}\hat{F}^{(r)}G^{(r)}\lambda_{ir}$ *and* $G^{(r)} = (T^{-1}F^{(r)\prime}\hat{F}^{(r)})^{-1}\left(N^{-1}\Lambda_r'\Lambda_r\right)^{-1}$.

Lemma A.6 *Under Assumptions A.1–A.4, we have*

$$\begin{aligned}
&\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\left[X_i^{(r)\prime}-\frac{1}{N}\sum_{k=1}^{N}a_{ik}^{(r)}X_k^{(r)\prime}\right]M_{\hat{F}^{(r)}}\varepsilon_i^{(r)}\\
=\;&\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\left[X_i^{(r)\prime}-\frac{1}{N}\sum_{k=1}^{N}a_{ik}^{(r)}X_k^{(r)\prime}\right]M_{F^{(r)}}\varepsilon_i^{(r)}+\sqrt{\frac{Th}{N}}\xi_{NT}^{(r)}\\
&+\sqrt{NTh}O_P((C_{NT}^{-1}+h)\|\hat{\delta}_r\|+\|\hat{\delta}_r\|^{3/2})+\sqrt{NTh}(Th)^{-1}O_P(\|\hat{\delta}_r\|^{1/2})+o_P(1),
\end{aligned}$$

where $\xi_{NT}^{(r)}=-\frac{1}{N}\sum_{i=1}^{N}\sum_{k=1}^{N}\frac{1}{T}(X_i^{(r)}-V_i^{(r)})'F^{(r)}[\frac{1}{T}F^{(r)\prime}F^{(r)}]^{-1}[\frac{1}{N}\Lambda_r'\Lambda_r]^{-1}\lambda_{kr}[\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}^{(r)}\varepsilon_{kt}^{(r)}]$ *and* $V_i^{(r)}=N^{-1}\sum_{k=1}^{N}a_{ik}^{(r)}X_k^{(r)}$.

Lemma A.7 *Under Assumptions A.1–A.4, we have*

$$\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\left[X_i^{(r)\prime}-\frac{1}{N}\sum_{k=1}^{N}a_{ik}^{(r)}X_k^{(r)\prime}\right]M_{\hat{F}^{(r)}}\Delta_i^{(r)}=\sqrt{NTh}B_{1\beta}^{(r)}+o_p(1),$$

where

$$\begin{aligned}
B_{1\beta}^{(r)}=&\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}k_{h,tr}E\left[(X_{it}-V_{it,r})A_{2,itr}\right](\frac{t-r}{T})^2\\
&+\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,tr}k_{h,sr}(X_{it}-V_{it,r})F_t'H^{(r)}[H^{(r)\prime}F_sA_{2,isr}+C_{1s}^{(r)}A_{1,isr}](\frac{s-r}{T})^2.
\end{aligned}$$

Lemma A.8 *Suppose that Assumptions A.1–A.5 hold. Then*

$$\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\left[X_i^{(r)\prime}-\frac{1}{N}\sum_{j=1}^{N}a_{ij}^{(r)}X_j^{(r)\prime}\right]M_{F^{(r)}}\varepsilon_i^{(r)}-\sqrt{\frac{N}{Th}}B_{2\beta}^{(r)}\xrightarrow{d}N\left(0,\Omega_r\right),$$

where $B_{2\beta}^{(r)}=(B_{2\beta,1}^{(r)},\cdots,B_{2\beta,P}^{(r)})'$ *with* $B_{2\beta,p}^{(r)}=-\frac{h}{N}tr\{P_{F^{(r)}}E_{\mathcal{C}}[\mathbf{X}_p^{(r)}\varepsilon^{(r)\prime}]\}$ *for* $p\in[P]$.

Lemma A.9 *Let* $\zeta_{NT}^{(r)}=-\frac{h}{N^2T}\sum_{i=1}^{N}\sum_{j=1}^{N}X_i^{(r)\prime}M_{\hat{F}^{(r)}}\varepsilon_j^{(r)}\varepsilon_j^{(r)\prime}\hat{F}^{(r)}(\frac{1}{T}F^{(r)\prime}\hat{F}^{(r)})^{-1}(\frac{1}{N}\Lambda_r'\Lambda_r)^{-1}\lambda_{ir}$. *Suppose that Assumptions A.1–A.4 hold. Then*

(i) $[D^{(r)}(\hat{F}^{(r)})]^{-1}-[D^{(r)}\left(F^{(r)}\right)]^{-1}=O_P(C_{NT}^{-1}+\|\hat{\delta}_r\|^{1/2})$;

(ii) $\sqrt{Th/N}(\xi_{NT}^{(r)}-B_{3\beta}^{(r)})=o_P(1)$;

(iii) $\sqrt{N/(Th)}(\zeta_{NT}^{(r)}-B_{4\beta}^{(r)})=o_P(1)$.

Lemma A.10 *Suppose that Assumptions A.1–A.4 hold. Then*

(i) $\frac{\sqrt{NTh}}{NT}tr\Big(P_{F^{(r)}}\left(\tilde{\mathbf{X}}_k^{(r)}\varepsilon^{(r)\prime}-E_{\mathcal{C}}[\tilde{\mathbf{X}}_k^{(r)}\varepsilon^{(r)\prime}]\right)\Big)=o_P(1)$

(ii) $\frac{\sqrt{NTh}}{NT}tr\Big(\tilde{\mathbf{X}}_k^{(r)}P_{\Lambda_r}\varepsilon^{(r)\prime}\Big)=o_P(1)$;

(iii) $\frac{\sqrt{NTh}}{NT}tr\Big(P_{F^{(r)}}\tilde{\mathbf{X}}_k^{(r)}P_{\Lambda_r}\varepsilon^{(r)\prime}\Big)=o_P(1)$.

Lemma A.11 *Suppose that Assumptions A.1–A.4 and A.7 hold. Then*

(i) $\max_t\left\|H^{(t)}H^{(t)\prime}-\left(F^{(t)\prime}F^{(t)}/T\right)^{-1}\right\|=O_P(h^2+(Th)^{-1}+N^{-1}\ln T)$;

(ii) $\max_t\left\|(\frac{1}{N}\hat{\Lambda}_t'\hat{\Lambda}_t)^{-1}-H^{(t)\prime}(\frac{1}{N}\Lambda_t'\Lambda_t)^{-1}H^{(t)}\right\|=O_P(h^2+C_{NT}^{-2}\ln T)$.

Proof of Proposition 3.1 (i) Let $S_{NT}(\beta_r,F^{(r)})=\frac{1}{NT}\sum_{i=1}^{N}(Y_i^{(r)}-X_i^{(r)}\beta_r)'M_{F^{(r)}}(Y_i^{(r)}-X_i^{(r)}\beta_r)$. Only in this proof will we use the superscript 0 to denote the true value. For example, $F_t^{(r)0}=k_{h,tr}^{*1/2}F_t^0$ and

$F^{(r)0\prime} = (F_1^{(r)0}, ..., F_T^{(r)0})'$, where F_t^0 denotes the true value of F_t. Similarly, $\Lambda_r^0 = (\lambda_{1r}^0, ..., \lambda_{Nr}^0)'$ and β_r^0 denote the true values of $\Lambda_r = (\lambda_{1r}, ..., \lambda_{Nr})'$ and β_r. Recall that $\Delta_{it}^{(r)} = k_{h,tr}^{*1/2}\Delta_i(t,r)$ and $\Delta_i^{(r)} = (\Delta_{i1}^{(r)}, ..., \Delta_{iT}^{(r)})'$. Let $\delta_r = \beta_r - \beta_r^0$ and $\delta_{F,r} = F^{(r)} - F^{(r)0}$. Noting that $Y_{it}^{(r)} = X_{it}^{(r)\prime}\beta_r^0 + \lambda_{ir}^{0\prime}F_t^{(r)0} + \Delta_i^{(r)}(t,r) + \varepsilon_{it}^{(r)}$, we have

$$
\begin{aligned}
& S_{NT}(\beta_r, F^{(r)}) - S_{NT}(\beta_r^0, F^{(r)0}) \\
&= \frac{1}{NT}\sum_{i=1}^N \left[-X_i^{(r)}\delta_r - \delta_{F,r}\lambda_{ir}^0 + \Delta_i^{(r)} + \varepsilon_i^{(r)}\right]' M_{F^{(r)}} \left[-X_i^{(r)}\delta_r - \delta_{F,r}\lambda_{ir}^0 + \Delta_i^{(r)} + \varepsilon_i^{(r)}\right] \\
&- \frac{1}{NT}\sum_{i=1}^N \left[\Delta_i^{(r)} + \varepsilon_i^{(r)}\right]' M_{F^{(r)0}} \left[\Delta_i^{(r)} + \varepsilon_i^{(r)}\right] \\
&= \frac{1}{NT}\sum_{i=1}^N \left[X_i^{(r)}\delta_r + \delta_{F,r}\lambda_{ir}^0\right]' M_{F^{(r)}} \left[X_i^{(r)}\delta_r + \delta_{F,r}\lambda_{ir}^0\right] - \frac{2}{NT}\sum_{i=1}^N \left[X_i^{(r)}\delta_r + \delta_{F,r}\lambda_{ir}^0\right]' M_{F^{(r)}} \left[\Delta_i^{(r)} + \varepsilon_i^{(r)}\right] \\
&+ \frac{1}{NT}\sum_{i=1}^N \left[\Delta_i^{(r)} + \varepsilon_i^{(r)}\right]' (P_{F^{(r)0}} - P_{F^{(r)}}) \left[\Delta_i^{(r)} + \varepsilon_i^{(r)}\right] \\
&= \tilde{S}_{NT}(\beta_r, F^{(r)}) - 2\delta_r' \frac{1}{NT}\sum_{i=1}^N X_i^{(r)\prime} M_{F^{(r)}}\varepsilon_i^{(r)} + \frac{2}{NT}\sum_{i=1}^N \lambda_{ir}^{0\prime} F^{(r)0\prime} M_{F^{(r)}}\varepsilon_i \\
&+ \frac{1}{NT}\sum_{i=1}^N \varepsilon_i^{(r)\prime}(P_{F^{(r)0}} - P_{F^{(r)}})\varepsilon_i^{(r)} + \frac{1}{NT}\sum_{i=1}^N \Delta_i^{(r)\prime}(P_{F^{(r)0}} - P_{F^{(r)}})\Delta_i^{(r)} - 2\delta_r' \frac{1}{NT}\sum_{i=1}^N X_i^{(r)\prime} M_{F^{(r)}}\Delta_i^{(r)} \\
&+ \frac{2}{NT}\sum_{i=1}^N \Delta_i^{(r)\prime} M_{F^{(r)}} F^{(r)0}\lambda_{ir} + \frac{2}{NT}\sum_{i=1}^N \Delta_i^{(r)\prime}(P_{F^{(r)0}} - P_{F^{(r)}})\varepsilon_i^{(r)},
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{S}_{NT}(\beta_r, F^{(r)}) &= \frac{1}{NT}\sum_{i=1}^N \left[X_i^{(r)}\delta_r + \delta_{F,r}\lambda_{ir}^0\right]' M_{F^{(r)}} \left[X_i^{(r)}\delta_r + \delta_{F,r}\lambda_{ir}^0\right] \\
&= \frac{1}{NT}\sum_{i=1}^N \delta_r' X_i^{(r)\prime} M_{F^{(r)}} X_i^{(r)}\delta_r + \operatorname{tr}\left[\frac{F^{(r)0\prime} M_{F^{(r)}} F^{(r)0}}{T}\frac{\Lambda_r^{0\prime}\Lambda_r^0}{N}\right] - \frac{2}{NT}\sum_{i=1}^N \delta_r' X_i^{(r)\prime} M_{F^{(r)}} F^{(r)0}\lambda_{ir}.
\end{aligned}
$$

By Lemma A.1, we have

$$
S_{NT}(\beta_r, F^{(r)}) = \tilde{S}_{NT}(\beta_r, F^{(r)}) + O_P(T^{-1/2} + N^{-1/4} + h)
$$

uniformly over bounded β_r and over $F^{(r)} \in \mathcal{F}$. Clearly, $\tilde{S}_{NT}(\beta_r^0, F^{(r)0}H^{(r)}) = 0$ because $M_{F^{(r)0}H^{(r)}}F^{(r)0} = M_{F^{(r)0}}F^{(r)0} = 0$. Following Bai (2009), we now show that $S_{NT}(\beta_r, F^{(r)}) > 0$ for any $(\beta_r, F^{(r)}) \neq (\beta_r^0, F^{(r)0}H^{(r)})$ where $H^{(r)}$ can be an arbitrary nonsingular matrix here. That is, $S_{NT}(\beta_r, F^{(r)})$ attains its unique minimum value 0 at $(\beta_r^0, F^{(r)0}H^{(r)})$. Denote

$$
\mathcal{A}^{(r)} = \frac{1}{NT}\sum_{i=1}^N X_i^{(r)\prime} M_{F^{(r)}} X_i^{(r)},\ \mathcal{B}^{(r)} = \frac{\Lambda_r'\Lambda_r}{N} \otimes \mathbb{I}_T,\ \mathcal{C}^{(r)} = \frac{1}{NT}\sum_{i=1}^N \lambda_{ir} \otimes M_{F^{(r)}} X_i^{(r)},\ \text{and } \eta^{(r)} = \operatorname{vec}(M_{F^{(r)}}F^{(r)0}).
$$

Then, we have

$$
\begin{aligned}
\tilde{S}_{NT}(\beta_r, F^{(r)}) &= \delta_r'\mathcal{A}^{(r)}\delta_r + \eta^{(r)\prime}\mathcal{B}^{(r)}\eta^{(r)} + 2\delta_r{}'\mathcal{C}^{(r)\prime}\eta^{(r)} \\
&= \delta_r'(\mathcal{A}^{(r)} - \mathcal{C}^{(r)\prime}\mathcal{B}^{(r)-1}\mathcal{C}^{(r)})\delta_r + \theta^{(r)\prime}\mathcal{B}^{(r)}\theta^{(r)} = \delta_r' D(F^{(r)})\delta_r + \theta^{(r)\prime}\mathcal{B}^{(r)}\theta^{(r)},
\end{aligned}
$$

where $\theta^{(r)} \equiv \eta^{(r)} + \mathcal{B}^{(r)-1}\mathcal{C}^{(r)}\delta_r$. By Assumptions A.1(i) and A.3(i), $\inf_{F^{(r)}\in\mathcal{F}} D(F^{(r)})$ and $\mathcal{B}^{(r)}$ are positive definite a.s. Thus $\tilde{S}_{NT}(\beta_r, F^{(r)}) \geq 0$. In addition, if either $\beta_r \neq \beta_r^0$ or $F^{(r)} \neq F^{(r)0}H^{(r)}$, then $\tilde{S}_{NT}(\beta_r, F^{(r)}) > 0$. Thus, $\tilde{S}_{NT}(\beta_r, F^{(r)})$ achieves its unique minimum at the true value $(\beta_r^0, F^{(r)0}H^{(r)})$. Furthermore, for any $\|\beta_r - \beta_r^0\| \geq c > 0$, we have $\tilde{S}_{NT}(\beta_r, F^{(r)0}) \geq \rho_{\min}^{(r)}c^2 > 0$, where $\rho_{\min}^{(r)} = \inf_{F^{(r)}\in\mathcal{F}} \mu_{\min}(D(F^{(r)}))$. This implies that $\hat{\beta}_r$ is consistent for β_r^0.

Note that the centered objective function satisfies $S_{NT}(\beta_r^0, F^{(r)0}H^{(r)}) = 0$ and by definition, $S_{NT}(\hat{\beta}_r, \hat{F}^{(r)}) \leq 0$. It follows that

$$0 \geq S_{NT}(\hat{\beta}_r, \hat{F}^{(r)}) = \tilde{S}_{NT}(\hat{\beta}_r, \hat{F}^{(r)}) + O_P(T^{-1/2} + N^{-1/4} + h). \tag{A.2}$$

It must be true that $\tilde{S}_{NT}(\hat{\beta}_r, \hat{F}^{(r)}) = O_P(T^{-1/2} + N^{-1/4} + h)$ and $\hat{\beta}_r - \beta_r^0 = O_P(T^{-1/4} + N^{-1/8} + h^{1/2})$.

(ii) (A.2) also implies that $\operatorname{tr}\big[(\frac{1}{T}F^{(r)0\prime}M_{\hat{F}^{(r)}}F^{(r)0})(\frac{1}{N}\Lambda_r'\Lambda_r)\big] = O_P(T^{-1/2} + N^{-1/4} + h)$. This, along with the fact that $\frac{1}{N}\Lambda_r'\Lambda_r = \Sigma_{\Lambda_r} + O(N^{-1/2})$ by Assumption A.3(i), implies that

$$\frac{1}{T}F^{(r)0\prime}M_{\hat{F}^{(r)}}F^{(r)0} = \frac{1}{T}F^{(r)0\prime}F^{(r)0} - \frac{1}{T}F^{(r)0\prime}\hat{F}^{(r)}\frac{1}{T}\hat{F}^{(r)\prime}F^{(r)0} = O_P(T^{-1/2} + N^{-1/4} + h). \tag{A.3}$$

By Assumption A.1(iii), we know $\frac{1}{T}F^{0\prime}F^0 \xrightarrow{p} \Sigma_F > 0$. With this, it is standard to show that $\frac{1}{T}F^{(r)0\prime}F^{(r)0} = \frac{1}{T}\sum_{t=1}^T k_{tr}^* F_t^0 F_t^{0\prime} = \Sigma_F + o_P(1)$ under Assumption A.1. It follows that

$$\begin{aligned} T^{-1}\hat{F}^{(r)\prime}P_{F^{(r)0}}\hat{F}^{(r)} &= T^{-1}\hat{F}^{(r)\prime}F^{(r)0}(T^{-1}F^{(r)0\prime}F^{(r)0})^{-1}F^{(r)0\prime}\hat{F}^{(r)} \\ &= T^{-1}\hat{F}^{(r)\prime}F^{(r)0}\left[\frac{1}{T}F^{(r)0\prime}\hat{F}^{(r)}\frac{1}{T}\hat{F}^{(r)\prime}F^{(r)0} + O_P(T^{-1/2} + N^{-1/4} + h)\right]^{-1} F^{(r)0\prime}\hat{F}^{(r)} \\ &= \mathbb{I}_R + O_P(T^{-1/2} + N^{-1/4} + h). \end{aligned}$$

Consequently, we have $\|P_{\hat{F}^{(r)}} - P_{F^{(r)0}}\|^2 = \operatorname{tr}\{(P_{\hat{F}^{(r)}} - P_{F^{(r)0}})^2\} = 2\operatorname{tr}(\mathbb{I}_R - T^{-1}\hat{F}^{(r)\prime}P_{F^{(r)0}}\hat{F}^{(r)}) = O_P(T^{-1/2} + N^{-1/4} + h)$. ■

Proof of Theorem 3.2. Noting that $Y_{it}^{(r)} = X_{it}^{(r)\prime}\beta_r + \lambda_{ir}'F_t^{(r)} + X_{it}^{(r)\prime}d_0(t,r) + d_i(t,r)'F_t^{(r)} + \varepsilon_{it}^{(r)} = X_{it}^{(r)\prime}\beta_r + \lambda_{ir}'F_t^{(r)} + \Delta_i^{(r)}(t,r) + \varepsilon_{it}^{(r)}$, we have

$$\begin{aligned} \hat{\beta}_r - \beta_r &= \left(\hat{D}^{(r)}\right)^{-1}\frac{1}{NT}\sum_{i=1}^N X_i^{(r)\prime}M_{\hat{F}^{(r)}}Y_i^{(r)} - \beta_r \\ &= \left(\hat{D}^{(r)}\right)^{-1}\frac{1}{NT}\left\{\sum_{i=1}^N X_i^{(r)\prime}M_{\hat{F}^{(r)}}F^{(r)}\lambda_{ir} + \sum_{i=1}^N X_i^{(r)\prime}M_{\hat{F}^{(r)}}\varepsilon_i^{(r)} + \sum_{i=1}^N X_i^{(r)\prime}M_{\hat{F}^{(r)}}\Delta_i^{(r)}\right\}. \end{aligned} \tag{A.4}$$

where $\hat{D}^{(r)} \equiv D^{(r)}(\hat{F}^{(r)}) = \frac{1}{NT}\sum_{i=1}^N X_i^{(r)\prime}M_{\hat{F}^{(r)}}X_i^{(r)}$. By Lemma A.5,

$$\begin{aligned} \frac{1}{NT}\sum_{i=1}^N X_i^{(r)\prime}M_{\hat{F}^{(r)}}F^{(r)}\lambda_{ir} &= \frac{1}{N^2T}\sum_{i=1}^N\sum_{j=1}^N X_i^{(r)\prime}M_{\hat{F}^{(r)}}X_j^{(r)}a_{ji}^{(r)}(\hat{\beta}_r - \beta_r) - \frac{1}{N^2T}\sum_{i=1}^N\sum_{j=1}^N a_{ij}^{(r)}X_i^{(r)\prime}M_{\hat{F}^{(r)}}\varepsilon_j^{(r)} \\ &\quad - \frac{1}{N^2T^2}\sum_{i=1}^N\sum_{j=1}^N X_i^{(r)\prime}M_{\hat{F}^{(r)}}\varepsilon_j^{(r)}\varepsilon_j^{(r)\prime}\hat{F}^{(r)}G^{(r)}\lambda_{ir} - \frac{1}{N^2T}\sum_{i=1}^N\sum_{j=1}^N X_i^{(r)\prime}M_{\hat{F}^{(r)}}\Delta_j^{(r)}a_{ji}^{(r)} \\ &\quad + O_P(C_{NT}^{-1} + \|\hat{\delta}_r\|^{1/2})O_P(\|\hat{\delta}_r\|) + O_P(C_{NT}^{-3} + h^4 + C_{NT}^{-1}h^3). \end{aligned}$$

Substituting this equation to (A.4), combining terms and multiplying by $\sqrt{NTh}$ yields

$$\hat{D}^{(r)}\sqrt{NTh}(\hat{\beta}_r - \beta_r)$$

$$= \frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\left[X_i^{(r)\prime}M_{\hat{F}^{(r)}} - \frac{1}{N}\sum_{j=1}^{N}a_{ij}^{(r)}X_j^{(r)\prime}M_{\hat{F}^{(r)}}\right]\varepsilon_i^{(r)} + \frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\left[X_i^{(r)\prime}M_{\hat{F}^{(r)}} - \frac{1}{N}\sum_{j=1}^{N}a_{ij}^{(r)}X_j^{(r)\prime}M_{\hat{F}^{(r)}}\right]\Delta_i^{(r)}$$

$$- \frac{\sqrt{NTh}}{N^2T^2}\sum_{i=1}^{N}\sum_{j=1}^{N}X_i^{(r)\prime}M_{\hat{F}^{(r)}}\varepsilon_j^{(r)}\varepsilon_j^{(r)\prime}\hat{F}^{(r)}G^{(r)}\lambda_{ir} + O_P(C_{NT}^{-1} + \|\hat{\delta}_r\|^{1/2})O_P(\|\hat{\delta}_r\|) + o_P(1), \quad (A.5)$$

where we use the fact that $\sqrt{NTh}(h^4 + C_{NT}^{-3} + C_{NT}^{-1}h^3) = o\,(1)$ under Assumption A.4(ii). By Lemma A.7,

$$\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\left[X_i^{(r)\prime} - \frac{1}{N}\sum_{k=1}^{N}a_{ik}^{(r)}X_k^{(r)\prime}\right]M_{\hat{F}^{(r)}}\Delta_i^{(r)} = \sqrt{NTh}B_{1\beta}^{(r)} + o_P(1). \quad (A.6)$$

Premultiplying both sides of (A.5) by $[\hat{D}^{(r)}]^{-1}$, using (A.6), and rearranging terms, we have

$$\sqrt{NTh}\left(\hat{\beta}_r - \beta_r - D(\hat{F}^{(r)})^{-1}B_{1\beta}^{(r)}\right) = [\hat{D}^{(r)}]^{-1}\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\left[X_i^{(r)\prime} - \frac{1}{N}\sum_{j=1}^{N}a_{ij}^{(r)}X_j^{(r)\prime}\right]M_{\hat{F}^{(r)}}\varepsilon_i^{(r)}$$
$$+ \sqrt{\frac{N}{Th}}[\hat{D}^{(r)}]^{-1}\zeta_{NT}^{(r)} + O_P(C_{NT}^{-1} + \|\hat{\delta}_r\|^{1/2})O_P(\|\hat{\delta}_r\|) + o_P(1).$$

By Lemma A.6, we have

$$\sqrt{NTh}\left(\hat{\beta}_r - \beta_r - [\hat{D}^{(r)}]^{-1}B_{1\beta}^{(r)}\right)$$
$$= [\hat{D}^{(r)}]^{-1}\left\{\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\left[X_i^{(r)\prime} - \frac{1}{N}\sum_{j=1}^{N}a_{ij}^{(r)}X_j^{(r)\prime}\right]M_{F^{(r)}}\varepsilon_i^{(r)} + \sqrt{\frac{Th}{N}}\xi_{NT}^{(r)} + \sqrt{\frac{N}{Th}}\zeta_{NT}^{(r)}\right\} + o_P(1).$$

By Lemma A.8,

$$\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\left[X_i^{(r)\prime} - \frac{1}{N}\sum_{j=1}^{N}a_{ij}^{(r)}X_j^{(r)\prime}\right]M_{F^{(r)}}\varepsilon_i^{(r)} - \sqrt{\frac{N}{Th}}B_{2\beta}^{(r)} \xrightarrow{d} N\left(0, \Omega_r\right).$$

By Lemma A.9(ii)-(iii), $\sqrt{Th/N}(\xi_{NT}^{(r)} - B_{3\beta}^{(r)}) = o_P(1)$ and $\sqrt{N/(Th)}(\zeta_{NT}^{(r)} - B_{4\beta}^{(r)}) = o_P(1)$. These results, along with the result in Lemma A.9(i) and Assumption A.2(i), implies that

$$\begin{aligned}
&\sqrt{NTh}\left(\hat{\beta}_r - \beta_r - \left[D^{(r)}\left(F^{(r)}\right)\right]^{-1}B_{1\beta}^{(r)}\right) - \left[D^{(r)}\left(F^{(r)}\right)\right]^{-1}\left[\sqrt{\frac{N}{Th}}B_{2\beta}^{(r)} + \sqrt{\frac{Th}{N}}B_{3\beta}^{(r)} + \sqrt{\frac{N}{Th}}B_{4\beta}^{(r)}\right] \\
=\;& \sqrt{NTh}\left(\hat{\beta}_r - \beta_r - \left[D^{(r)}\left(F^{(r)}\right)\right]^{-1}[B_{1\beta}^{(r)} + \frac{1}{Th}B_{2\beta}^{(r)} + \frac{1}{N}B_{3\beta}^{(r)} + \frac{1}{Th}B_{4\beta}^{(r)}]\right) \\
=\;& \left[D^{(r)}\left(F^{(r)}\right)\right]^{-1}\left\{\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\left[X_i^{(r)\prime} - \frac{1}{N}\sum_{j=1}^{N}a_{ij}^{(r)}X_j^{(r)\prime}\right]M_{F^{(r)}}\varepsilon_i^{(r)} - \sqrt{\frac{N}{Th}}B_{2\beta}^{(r)}\right\} + o_P\left(1\right) \\
&\xrightarrow{d} N\left(0,\; D_0^{(r)-1}\Omega_r D_0^{(r)-1}\right).
\end{aligned}$$

This complete the proof of Theorem 3.2. ■

Proof of Theorem 3.3. (i) Recall that $\hat{\delta}_r = \hat{\beta}_r - \beta_r$. Noting that $Y_s^{(r)} - X_s^{(r)}\hat{\beta}_r = -X_s^{(r)}\hat{\delta}_r + \Lambda_r F_s^{(r)} + \Delta_s^{(r)} + \varepsilon_s^{(r)}$, $\Delta_s^{(r)} = X_s^{(r)}D_0(s,r) + D(s,r)F_s^{(r)}$, we start with $(NT)^{-1}(Y^{(r)} - X^{(r)}\hat{\beta}_r)(Y^{(r)} - X^{(r)}\hat{\beta}_r)'\hat{F}^{(r)}\hat{V}_{NT}^{(r)-1} = \hat{F}^{(r)}$

to obtain the following decomposition for $\hat{F}_t^{(r)} - H^{(r)\prime} F_t^{(r)} - B_t^{(r)}$ as follows

$$
\begin{aligned}
&\hat{F}_t^{(r)} - H^{(r)\prime} F_t^{(r)} - B_t^{(r)} \\
&= \hat{V}_{NT}^{(r)-1} \frac{1}{NT} \sum_{s=1}^{T} \hat{F}_s^{(r)} \left[-X_s^{(r)} \hat{\delta}_r + \Lambda_r F_s^{(r)} + \Delta_s^{(r)} + \varepsilon_s^{(r)} \right]' \left[-X_t^{(r)} \hat{\delta}_r + \Lambda_r F_t^{(r)} + \Delta_t^{(r)} + \varepsilon_t^{(r)} \right] - H^{(r)\prime} F_t^{(r)} - B_t^{(r)} \\
&= \hat{V}_{NT}^{(r)-1} \left\{ \frac{1}{T} \sum_{s=1}^{T} \hat{F}_s^{(r)} E(\varepsilon_s^{(r)\prime} \varepsilon_t^{(r)}/N) + \frac{1}{T} \sum_{s} \hat{F}_s^{(r)} \left[\varepsilon_s^{(r)\prime} \varepsilon_t^{(r)}/N - E(\varepsilon_s^{(r)\prime} \varepsilon_t^{(r)}/N) \right] + \frac{1}{T} \sum_{s=1}^{T} \hat{F}_s^{(r)} F_s^{(r)\prime} \Lambda_r' \varepsilon_t^{(r)}/N \right. \\
&\quad + \frac{1}{T} \sum_{s=1}^{T} \hat{F}_s^{(r)} F_t^{(r)\prime} \Lambda_r' \varepsilon_s^{(r)}/N + \left[\frac{1}{TN} \sum_{s=1}^{T} \hat{F}_s^{(r)} \Delta_s^{(r)\prime} \Lambda_r F_t^{(r)} - \hat{V}_{NT}^{(r)} B_{2t}^{(r)} \right] + \frac{1}{TN} \sum_{s=1}^{T} \hat{F}_s^{(r)} \Delta_s^{(r)\prime} \Delta_t^{(r)} \\
&\quad + \frac{1}{TN} \sum_{s=1}^{T} \hat{F}_s^{(r)} \Delta_s^{(r)\prime} \varepsilon_t^{(r)} + \frac{1}{TN} \sum_{s=1}^{T} \hat{F}_s^{(r)} \varepsilon_s^{(r)\prime} \Delta_t^{(r)} + \left[\frac{1}{TN} \sum_{s=1}^{T} \hat{F}_s^{(r)} F_s^{(r)\prime} \Lambda_r' \Delta_t^{(r)} - \hat{V}_{NT}^{(r)} B_{1t}^{(r)} \right] \\
&\quad + \frac{1}{NT} \sum_{s=1}^{T} \hat{F}_s^{(r)} \hat{\delta}_r' X_s^{(r)\prime} X_t^{(r)} \hat{\delta}_r - \frac{1}{NT} \sum_{s=1}^{T} \hat{F}_s^{(r)} \hat{\delta}_r' X_s^{(r)\prime} \Lambda_r F_t^{(r)} - \frac{1}{NT} \sum_{s=1}^{T} \hat{F}_s^{(r)} \hat{\delta}_r' X_s^{(r)\prime} \Delta_t^{(r)} - \frac{1}{NT} \sum_{s=1}^{T} \hat{F}_s^{(r)} \hat{\delta}_r' X_s^{(r)\prime} \varepsilon_t^{(r)} \\
&\quad \left. - \frac{1}{NT} \sum_{s=1}^{T} \hat{F}_s^{(r)} F_s^{(r)\prime} \Lambda_r' X_t^{(r)} \hat{\delta}_r - \frac{1}{NT} \sum_{s=1}^{T} \hat{F}_s^{(r)} \Delta_s^{(r)\prime} X_t^{(r)} \hat{\delta}_r - \frac{1}{NT} \sum_{s=1}^{T} \hat{F}_s^{(r)} \varepsilon_s^{(r)\prime} X_t^{(r)} \hat{\delta}_r \right\} \\
&\equiv \sum_{l=1}^{16} A_l(t,r), \qquad \text{(A.7)}
\end{aligned}
$$

where we use the fact that $B_t^{(r)} = B_{1t}^{(r)} + B_{2t}^{(r)}$. The first nine terms do not depend on $\hat{\delta}_r$ explicitly and they have the same expressions as those in Su and Wang (2017, 2020). Following Su and Wang (2017, 2020), we can also show that $\sqrt{Nh} A_j(t,r) = o_P(1)$ for $j = 1, 2, 4, 5, 6, 7, 8, 9$, and that $\sqrt{Nh} A_3(t,r)$ determines the asymptotic distribution. $A_5(t,r)$ and $A_9(t,r)$ are associated with the bias, which we focus on below.

For $A_5(t,r)$, noting that $\hat{V}_{NT}^{(r)}$ is asymptotically nonsingular by Lemma A.2(i), it suffices to show that $\bar{A}_5(t,r) \equiv \hat{V}_{NT}^{(r)} A_5(t,r) = o_P\left((Nh)^{-1/2}\right)$. Let $B_{2t,1}^{(r)}$ and $B_{2t,2}^{(r)}$ be as defined in (A.1). Let $\varphi_{1,tr} = \hat{F}_t^{(r)} - H^{(r)\prime} F_t^{(r)} - B_t^{(r)}$, $\varphi_{2,tr} = H^{(r)\prime} F_t^{(r)}$, $\varphi_{3,tr} = B_t^{(r)}$, $\chi_{21,t}^{(r)} = 0$, $\chi_{22,t}^{(r)} = \hat{V}_{NT}^{(r)} B_{2t,1}^{(r)}$, and $\chi_{23,t}^{(r)} = \hat{V}_{NT}^{(r)} B_{2t,2}^{(r)}$. For $\bar{A}_5(t,r)$, we make the following decomposition

$$
\bar{A}_5(t,r) = \frac{1}{TN} \sum_{s=1}^{T} \hat{F}_s^{(r)} \Delta_s^{(r)\prime} \Lambda_r F_t^{(r)} - \hat{V}_{NT}^{(r)} B_{2t}^{(r)} = \sum_{l=1}^{3} \left[\frac{1}{TN} \sum_{s} \varphi_{l,sr} \Delta_s^{(r)\prime} \Lambda_r F_t^{(r)} - \chi_{2l,t}^{(r)} \right] \equiv \sum_{l=1}^{3} \bar{A}_{5,l}(t,r).
$$

Following the proof of Lemma A.3 in Su and Wang (2020) and Theorem 3.2, we can show that $\sqrt{Nh} \bar{A}_{5,1}(t,r) = \sqrt{Nh} O_P(C_{NT}^{-2} + \|\hat{\delta}_r\|) = o_P(1)$ as Theorem 3.2 implies that $\hat{\delta}_r = O_P(N^{-1} + (Th)^{-1}) = o_P((Nh)^{-1/2})$. For $\bar{A}_{5,2}(t,r)$, noting that $[H^{(r)\prime}]^{-1} \hat{V}_{NT}^{(r)} B_{2t,1}^{(r)} = N^{-1} E(F_t A_{2,tr}') \Lambda_r F_t k_{h,tr}^{*1/2} h^2 \kappa_2 = N^{-1} E(F_t A_{2,tr}') \Lambda_r F_t^{(r)} h^2 \kappa_2$, we have

$$
\begin{aligned}
\bar{A}_{5,2}(t,r) &= \frac{1}{TN} \sum_{s=1}^{T} H^{(r)\prime} F_s^{(r)} \Delta_s^{(r)\prime} \Lambda_r F_t^{(r)} - \hat{V}_{NT}^{(r)} B_{2t,1}^{(r)} \\
&= H^{(r)\prime} \frac{1}{TN} \sum_{s=1}^{T} \sum_{i=1}^{N} k_{h,sr}^* F_s A_{1,isr} (\frac{s-r}{T}) \lambda_{ir}' F_t^{(r)} \\
&\quad + H^{(r)\prime} \left[\frac{1}{TN} \sum_{s=1}^{T} \sum_{i=1}^{N} k_{h,sr}^* F_s A_{2,isr} (\frac{s-r}{T})^2 \lambda_{ir} - N^{-1} E(F_t A_{2,tr}') \Lambda_r h^2 \kappa_2 \right] F_t^{(r)} + O_P\left(h^3\right) \\
&\equiv H^{(r)\prime} \bar{A}_{5,21}(t,r) F_t^{(r)} + H^{(r)\prime} \bar{A}_{5,22}(t,r) F_t^{(r)} + O_P\left(h^3\right).
\end{aligned}
$$

For $\bar{A}_{5,21}(t,r)$, we have

$$\begin{aligned}\bar{A}_{5,21}(t,r) &= \frac{1}{TN}\sum_{s=1}^{T}\sum_{i=1}^{N}k_{h,sr}^{*}F_sX_{is}'\beta_r^{(1)}\lambda_{ir}'\frac{s-r}{T}+\frac{1}{TN}\sum_{s=1}^{T}\sum_{i=1}^{N}k_{h,sr}^{*}F_sF_s'\lambda_{ir}^{(1)}\lambda_{ir}'\frac{s-r}{T}\\ &= \frac{1}{TN}\sum_{s=1}^{T}\sum_{i=1}^{N}k_{h,sr}^{*}F_sX_{is}'\beta_r^{(1)}\lambda_{ir}'\frac{s-r}{T}+\frac{1}{T}\sum_{s=1}^{T}k_{h,sr}^{*}F_sF_s'\frac{s-r}{T}(N^{-1}\Lambda_i^{(1)\prime}\Lambda_i)\equiv\sum_{\ell=1}^{2}\bar{A}_{5,21\ell}(t,r).\end{aligned}$$

As in Su and Wang (2020), $\bar{A}_{5,212}(t,r)=O_P((Th)^{-1/2}\,h+(Th)^{-1})=o_P((Nh)^{-1/2})$ by the fact that $\frac{1}{T}\sum_s k_{h,sr}^{*}F_sF_s'$ $(\frac{s-r}{T})=O_P((Th)^{-1/2}\,h)+O((Th)^{-1})$. For $\bar{A}_{5,211}(t,r)$, we have

$$\begin{aligned}\bar{A}_{5,211}(t,r) &= \frac{1}{NT}\sum_{s=1}^{T}\sum_{i=1}^{N}\frac{s-r}{T}k_{h,sr}^{*}\left[F_sX_{is}'-E(F_sX_{is}')\right]\beta_r^{(1)}\lambda_{ir}'+\frac{1}{N}\sum_{i=1}^{N}\left[\frac{1}{T}\sum_{s=1}^{T}\frac{s-r}{T}k_{h,sr}^{*}E(F_sX_{is}')\right]\beta_r^{(1)}\lambda_{ir}'\\ &= O_P((NTh)^{-1/2}\,h)+O((Th)^{-1})=o_P((Nh)^{-1/2}),\end{aligned}$$

where we use the fact that $\frac{1}{T}\sum_{s=1}^{T}\frac{s-r}{T}k_{h,sr}^{*}E(F_sX_{is}')=\frac{1}{T}\sum_{s=1}^{T}\frac{s-r}{T}k_{h,sr}^{*}\Sigma_{FX}=O((Th)^{-1})$ by the property of Riemann integral for interior point r. Then $\bar{A}_{5,21}(t,r)=o_P((Nh)^{-1/2})$. Note that

$$\begin{aligned}&\frac{1}{TN}\sum_{s=1}^{T}\sum_{i=1}^{N}k_{h,sr}^{*}F_sA_{2,isr}(\frac{s-r}{T})^2\lambda_{ir}-N^{-1}E(F_tA_{2,tr}')\Lambda_rF_t^{(r)}h^2\kappa_2\\ =\ &\frac{1}{2TN}\sum_{s=1}^{T}\sum_{i=1}^{N}k_{h,sr}^{*}F_s[X_{is}'\beta_r^{(2)}+F_s'\lambda_{ir}^{(2)}](\frac{s-r}{T})^2\lambda_{ir}-\frac{1}{2N}E\left[F_t(\beta_r^{(2)}X_t'+F_t'\Lambda_r^{(2)\prime})\right]\Lambda_rh^2\kappa_2\\ =\ &\frac{1}{2N}\sum_{s=1}^{T}(\frac{s-r}{T})^2k_{h,sr}^{*}F_s\beta_r^{(2)}\frac{X_s'\Lambda_r}{N}-\frac{1}{2N}E[F_t\beta_r^{(2)}\frac{X_t'\Lambda_r}{N}]h^2\kappa_2\\ &+\frac{1}{2}\left[\frac{1}{T}\sum_{s=1}^{T}(\frac{s-r}{T})^2k_{h,sr}^{*}F_sF_s'-E[F_tF_t']h^2\kappa_2\right](\frac{1}{N}\Lambda_r^{(2)\prime}\Lambda_r).\end{aligned}$$

It is easy to show that either term on the right hand side of the above equation is $O_P\left((Th)^{-1/2}h^2+(Th)^{-1}+h^3\right)$ $=o_P((Nh)^{-1/2})$. Thus $\bar{A}_{5,22}(t,r)=o_P((Nh)^{-1/2})$ and $\bar{A}_{5,2}(t,r)=o_P((Nh)^{-1/2})$.

Note that $\Delta_s^{(r)}=X_s^{(r)}d_0\left(s,r\right)+D\left(s,r\right)F_s^{(r)}=(\Delta_{1s}^{(r)},...,\Delta_{Ns}^{(r)})'$, and

$$\hat{V}_{NT}^{(r)}B_{2t,2}^{(r)}=\frac{1}{TN}\sum_{s}(\frac{s-r}{T})^2k_{h,sr}^{*}E[\bar{C}_{1s}^{(r)}A_{1,sr}']\Lambda_rF_t^{(r)}\equiv\alpha_1^{(r)}F_t^{(r)},$$

It follows that

$$\bar{A}_{5,3}(t,r)=\frac{1}{TN}\sum_{s}B_s^{(r)}\Delta_s^{(r)\prime}\Lambda_rF_t^{(r)}-\hat{V}_{NT}^{(r)}B_{2t,2}^{(r)}=\left[\frac{1}{TN}\sum_{s}\sum_{i}B_s^{(r)}\Delta_{is}^{(r)}\lambda_{ir}'-\alpha_1^{(r)}\right]F_t^{(r)}\equiv\bar{A}_{5,3}(r)F_t^{(r)}.$$

For $\bar{A}_{5,3}(r)$, we have

$$\begin{aligned}\bar{A}_{5,31}(r) &= \frac{1}{TN}\sum_{s}\sum_{i}B_s^{(r)}\Delta_{is}^{(r)}\lambda_{ir}'-\alpha_1^{(r)}=\frac{1}{TN}\sum_{s}\sum_{i}(\frac{s-r}{T})^2k_{h,sr}^{*}C_{1s}^{(r)}A_{1,isr}\lambda_{ir}'-\alpha_{1,1}^{(r)}+O_P\left(h^3\right)\\ &= \frac{1}{TN}\sum_{s}\sum_{i}(\frac{s-r}{T})^2k_{h,sr}^{*}\bar{C}_{1s}^{(r)}A_{1,isr}\lambda_{ir}'-\alpha_{1,1}^{(r)}+O_P\left(h^3\right)\\ &= \frac{1}{TN}\sum_{s}(\frac{s-r}{T})^2k_{h,sr}^{*}(\bar{C}_{1s}^{(r)}A_{1,sr}'-E[\bar{C}_{1s}^{(r)}A_{1,sr}']\}\Lambda_r+O_P\left(h^3\right)\end{aligned}$$

$$= \quad O_P((N/h)^{-1/2}h^2) + O_P\left(h^3\right) = o_P(Nh)^{-1/2}.$$

In sum, we have shown that $\sqrt{Nh}\bar{A}_5(t,r) = o_P(1)$.

For $A_9(t,r)$, we want to show that $\bar{A}_9(t,r) \equiv \hat{V}_{NT}^{(r)} A_9(t,r) = o_P(Nh)^{-1/2}$. Note that

$$\bar{A}_9(t,r) = \frac{1}{TN}\sum_{s=1}^{T} \hat{F}_s^{(r)} F_s^{(r)\prime}\Lambda_r'\Delta_t^{(r)} - \hat{V}_{NT}^{(r)} B_{1t}^{(r)} = \sum_{l=1}^{3}\left[\frac{1}{TN}\sum_s \varphi_{l,sr}\Delta_s^{(r)\prime}\Lambda_r F_t^{(r)} - \chi_{1l,t}^{(r)}\right] \equiv \sum_{l=1}^{3}\bar{A}_{9,l}(t,r),$$

where $\chi_{11,t} = 0$, $\chi_{12,t} = \hat{V}_{NT}^{(r)} B_{1t}^{(r)}$ and $\chi_{13,t} = 0$. As in Su and Wang (2020), we can readily show that

$$\begin{aligned} \sqrt{Nh}\bar{A}_{9,1}(t,r) &= \sqrt{Nh}O_P(C_{NT}^{-1} + \|\hat{\delta}_r\|^{1/2})O_P(h^{1/2}) = o_P(1) \text{ and} \\ \sqrt{Nh}\bar{A}_{9,3}(t,r) &= \sqrt{Nh}O_P((Th)^{-1/2}h^{3/2} + (Th)^{-1}h^{1/2} + h^{5/2}) = o_P(1) \end{aligned}$$

under the conditions that $Nh/T \to 0$ and $Nh^6 \to 0$ in Assumption A.4(ii). Noting that $\hat{V}_{NT}^{(r)} B_{1t}^{(r)} = \hat{V}_{NT}^{(r)}[k_{h,tr}^{*1/2} C_{1t}^{(r)} \frac{t-r}{T}$ $+ k_{h,tr}^{*1/2} C_{3t}^{(r)} (\frac{t-r}{T})^2] = \alpha_{2,1} + \alpha_{2,2}$ with $\alpha_{2,l} = N^{-1} k_{h,tr}^{*1/2} H^{(r)\prime}\Sigma_F \Lambda_r' A_{l,tr} (\frac{t-r}{T})^l$ for $l = 1, 2$, we have

$$\begin{aligned} \bar{A}_{9,2}(t,r) &= \frac{1}{TN}\sum_{s=1}^{T} H^{(r)\prime} F_s^{(r)} F_s^{(r)\prime}\Lambda_r'\Delta_t^{(r)} - \hat{V}_{NT}^{(r)} B_{1t}^{(r)} = H^{(r)\prime}\frac{1}{TN}\sum_{s=1}^{T} k_{h,sr}^{*} F_s F_s'\Lambda_r'\Delta_t^{(r)} - (\alpha_{2,1} + \alpha_{2,2}) \\ &= H^{(r)\prime}\frac{1}{N}\Sigma_F\Lambda_r'\Delta_t^{(r)} - (\alpha_{2,1} + \alpha_{2,2}) + O_P((Th)^{-1/2}h) \\ &= H^{(r)\prime}\Sigma_F\left[\frac{1}{N}\sum_{i=1}^{N}\lambda_{ir}\left(X_{it}'\beta_r^{(1)} + \lambda_{ir}^{(1)\prime}F_t\right) - \frac{1}{N}\Lambda_r' A_{1,tr}\right]\frac{t-r}{T}k_{h,tr}^{*1/2} \\ &+ \frac{1}{2}H^{(r)\prime}\Sigma_F\left[\frac{1}{N}\sum_{i=1}^{N}\lambda_{ir}\left(X_{it}'\beta_r^{(2)} + \lambda_{ir}^{(2)\prime}F_t\right) - \frac{1}{N}\Lambda_r' A_{2,tr}\right](\frac{t-r}{T})^2 k_{h,tr}^{*1/2} + O_P((Th)^{-1/2}h + h^3) \\ &= O_P((Th)^{-1/2}h + h^3) = o_P(Nh)^{-1/2}. \end{aligned}$$

Then $\sqrt{Nh}A_9(t,r) = o_P(1)$.

In summary, the bias term $B_t^{(r)}$ is

$$\begin{aligned} B_t^{(r)} &= B_{1t}^{(r)} + B_{2t,1}^{(r)} + B_{2t,2}^{(r)} \\ &= \left[k_{h,tr}^{*1/2} C_{1t}^{(r)}\frac{t-r}{T} + k_{h,tr}^{*1/2} C_{3t}^{(r)}(\frac{t-r}{T})^2\right] + \hat{V}_{NT}^{(r)-1} H^{(r)\prime} N^{-1} E(F_t A_{2,tr}')\Lambda_r F_t^{(r)} h^2\kappa_2 \\ &\quad + \hat{V}_{NT}^{(r)-1}\frac{1}{TN}\sum_s(\frac{s-r}{T})^2 k_{h,sr}^{*} E[\bar{C}_{1s}^{(r)} A_{1,sr}']\Lambda_r F_t^{(r)} \\ &= k_{h,tr}^{*1/2}\left[C_{1t}^{(r)}\frac{t-r}{T} + C_{2t}^{(r)}(\frac{t-r}{T})^2 + h^2\kappa_2 C_{3t}^{(r)} + h^2\kappa_2 C_{4t}^{(r)}\right], \end{aligned}$$

where $C_{lt}^{(r)}$, $l \in [4]$, are as defined in (3.1).

The terms $A_{10}(t,r)$ to $A_{17}(t,r)$ are either $O_P(\|\hat{\delta}_r\|)$ or $o_P(\|\hat{\delta}_r\|)$. This, in conjunction with the fact that $\hat{\delta}_r = O_P(N^{-1} + (Th)^{-1})$ by Theorem 3.2, implies that $\sqrt{Nh}\sum_{i=10}^{17} A_i(t,r) = o_P(1)$. In addition, Su and Wang (2017, 2020) have shown that $K_r^*(\frac{t-r}{Th})^{-1/2}\sqrt{Nh}A_3(t,r) \xrightarrow{d} N(0, V_r^{-1}Q_r\Gamma_{rt}Q_r'V_r^{-1})$. Combining these results yields the desired result in Theorem 3.3(i).

(ii) Let $\Upsilon^{(r)} = \hat{F}^{(r)} - F^{(r)}H^{(r)} - B^{(r)}$. Recall that $\Delta_i^{(r)} = (\Delta_{i,1}^{(r)}, ..., \Delta_{i,T}^{(r)})'$ with $\Delta_{i,t}^{(r)} = k_{h,tr}^{*1/2}\Delta_i(t,r)$. Noting that $\hat{\lambda}_{ir} = \frac{1}{T}\hat{F}^{(r)\prime}(Y_i^{(r)} - X_i^{(r)}\hat{\beta}_r)$ and $Y_i^{(r)} - X_i^{(r)}\hat{\beta}_r = -X_i^{(r)}\hat{\delta}_r + F^{(r)}\lambda_{ir} + \Delta_i^{(r)} + \varepsilon_i^{(r)}$, we have

$$\begin{aligned}
\hat{\lambda}_{ir} - H^{(r)-1}\lambda_{ir} &= \frac{1}{T}\hat{F}^{(r)\prime}\left[-X_i^{(r)}\hat{\delta}_r + F^{(r)}\lambda_{ir} + \Delta_i^{(r)} + \varepsilon_i^{(r)}\right] - H^{(r)-1}\lambda_{ir} \\
&= \frac{1}{T}H^{(r)\prime}F^{(r)\prime}\varepsilon_i^{(r)} + \frac{1}{T}\Upsilon^{(r)\prime}\varepsilon_i^{(r)} - \frac{1}{T}\hat{F}^{(r)\prime}\Upsilon^{(r)}H^{(r)-1}\lambda_{ir} + \frac{1}{T}\Upsilon^{(r)\prime}\Delta_i^{(r)} \\
&\quad + \frac{1}{T}H^{(r)\prime}F^{(r)\prime}\Delta_i^{(r)} + \frac{1}{T}B^{(r)\prime}\varepsilon_i^{(r)} - \frac{1}{T}\hat{F}^{(r)\prime}B^{(r)}H^{(r)-1}\lambda_{ir} + \frac{1}{T}B^{(r)\prime}\Delta_i^{(r)} \\
&\quad - \frac{1}{T}\Upsilon^{(r)\prime}X_i^{(r)}\hat{\delta}_r - \frac{1}{T}H^{(r)\prime}F^{(r)\prime}X_i^{(r)}\hat{\delta}_r - \frac{1}{T}B^{(r)\prime}X_i^{(r)}\hat{\delta}_r \equiv \sum\nolimits_{l=1}^{11} D_l(i,r), \qquad \text{(A.8)}
\end{aligned}$$

Su and Wang (2017, 2020) have studied the terms $D_1(i,r)$ to $D_8(i,r)$. The only difference for these terms is that

$$\begin{aligned}
\Delta_{i,t}^{(r)} &= k_{h,tr}^{*1/2}\left\{X_{it}^{\prime}[\beta(\frac{t}{T}) - \beta(\frac{r}{T})] + F_t^{\prime}[\lambda_i(\frac{t}{T}) - \lambda_i(\frac{r}{T})]\right\} \\
&= k_{h,tr}^{*1/2}\left\{[X_{it}^{\prime}\beta_r^{(1)} + F_t^{\prime}\lambda_{ir}^{(1)}]\frac{t-r}{T} + \frac{1}{2}\left[X_{it}^{\prime}\beta_r^{(2)} + F_t^{\prime}\lambda_{ir}^{(2)}\right]\left(\frac{t-r}{T}\right)^2 + O_P\left(\left(\frac{t-r}{T}\right)^3\right)\right\}
\end{aligned}$$

in this paper, instead of $\Delta_{i,t}^{(r)} = k_{h,tr}^{*1/2}F_t^{\prime}[\lambda_i(\frac{t}{T}) - \lambda_i(\frac{r}{T})]$ in Su and Wang (2020). Following the analyses in Su and Wang (2017, 2020), we can show that $\sqrt{Th}D_l(i,r) = o_P(1)$ for $l = 2, 3, 4, 6$, while $D_5(i,r)$, $D_7(i,r)$ and $D_8(i,r)$ contribute to the asymptotic bias. Following Su and Wang (2017, 2020), we can show that

$$D_5(i,r) = \frac{1}{T}H^{(r)\prime}F^{(r)\prime}\Delta_i^{(r)} = h^2\kappa_2 H^{(r)\prime}E\left(F_t A_{2,itr}\right) + O_P((Th)^{-1/2}h + h^3).$$

For $D_7(i,r)$, we have

$$-D_7(i,r) = \frac{1}{T}\sum_{t=1}^{T}\Upsilon_t^{(r)}B_t^{(r)\prime}H^{(r)-1}\lambda_{ir} + \frac{1}{T}\sum_{t=1}^{T}H^{(r)\prime}F_t^{(r)}B_t^{(r)\prime}H^{(r)-1}\lambda_{ir} + \frac{1}{T}\sum_{t=1}^{T}B_t^{(r)}B_t^{(r)\prime}H^{(r)-1}\lambda_{ir} = \sum\nolimits_{l=1}^{3} D_{7,l}(i,r),$$

where $\Upsilon_t^{(r)} = \hat{F}_t^{(r)} - H^{(r)\prime}F_t^{(r)} - B_t^{(r)}$. Note that when $r \in [\lfloor Th \rfloor, T - \lfloor Th \rfloor]$, we have

$$\begin{aligned}
\|D_{7,1}(i,r)\| &\lesssim \left\{\frac{1}{T}\sum_{t=1}^{T}\left\|\Upsilon_t^{(r)}\right\|^2\right\}^{1/2}\left\{\frac{1}{T}\sum_{t=1}^{T}\left\|B_t^{(r)}\right\|^2\right\}^{1/2} = O_P(C_{NT}^{-1} + \|\hat{\delta}_r\|)O_P(h), \\
D_{7,2}(i,r) &= \frac{1}{T}\sum_{t=1}^{T}k_{h,tr}^{*}H^{(r)\prime}F_t\left[C_{1t}^{(r)}\frac{t-r}{T} + C_{2t}^{(r)}(\frac{t-r}{T})^2 + (C_{3t}^{(r)} + C_{4t}^{(r)})\kappa_2 h^2\right]^{\prime}H^{(r)-1}\lambda_{ir} \\
&= \kappa_2 h^2 H^{(r)\prime}E\left[F_t(\bar{C}_{2t}^{(r)} + \bar{C}_{3t}^{(r)} + \bar{C}_{4t}^{(r)})^{\prime}\right]H^{(r)-1}\lambda_{ir} + O_P((Th)^{-1/2}h),
\end{aligned}$$

$$\begin{aligned}
D_{7,3} &= \frac{1}{T}\sum_{t=1}^{T}\left[C_{1t}^{(r)}\frac{t-r}{T} + C_{2t}^{(r)}\kappa_2 h^2 + C_{3t}^{(r)}(\frac{t-r}{T})^2\right]\left[C_{1t}^{(r)}\frac{t-r}{T} + C_{2t}^{(r)}\kappa_2 h^2 + C_{3t}^{(r)}(\frac{t-r}{T})^2\right]^{\prime}H^{(r)-1}\lambda_{ir} \\
&= \kappa_2 h^2 E(\bar{C}_{1t}^{(r)}\bar{C}_{1t}^{(r)\prime})H^{(r)-1}\lambda_{ir} + O_P((Th)^{-1/2}h + h^3).
\end{aligned}$$

and thus $D_7(i,r) = \kappa_2 h^2\left\{H^{(r)\prime}E[F_t(\bar{C}_{2t}^{(r)} + \bar{C}_{3t}^{(r)} + \bar{C}_{4t}^{(r)})^{\prime}] + E(\bar{C}_{1t}^{(r)}\bar{C}_{1t}^{(r)\prime})\right\}H^{(r)-1}\lambda_{ir} + O_P(T^{-1/2}h^{3/2} + h^3)$. When $r \in [1, \lfloor Th \rfloor) \cup (T - \lfloor Th \rfloor, T]$, it is easy to show that the leading bias term is contributed by $\frac{1}{T}\sum_{t=1}^{T}k_{h,tr}^{*}H^{(r)\prime}E(F_t\bar{C}_{1t}^{(r)\prime})\frac{t-}{T}$
$O(h)$.

Similarly,

$$D_8(i,r) = \frac{1}{T}\sum_{t=1}^{T}\left[C_{1t}^{(r)}\frac{t-r}{T} + C_{2t}^{(r)}(\frac{t-r}{T})^2 + (C_{3t}^{(r)} + C_{4t}^{(r)})\kappa_2 h^2\right]\left[X_{it}'\beta_r^{(1)} + \lambda_{ir}^{(1)\prime}F_t\right]k_{h,tr}^*\frac{t-r}{T} + O_P(h^3)$$
$$= \kappa_2 h^2 E(\bar{C}_{1t}^{(r)}A_{1,itr}) + O_P((Th)^{-1/2}h^2 + h^3).$$

In addition, noting that $D_l(i,r) = o_P(\|\hat{\delta}_r\|)$ for $l = 9, 11$ and $D_{10}(i,r) = O_P(\|\hat{\delta}_r\|)$, we have $\sqrt{Th}\sum_{l=9}^{11} D_l(i,r) = O_P(\|\hat{\delta}_r\|) = o_P(1)$. In sum, we have shown that

$$\sum_{l=2}^{11} D_l(i,r) = \kappa_2 h^2\left[E(\bar{C}_{1t}^{(r)}A_{1,itr}) + H^{(r)\prime}E\left(F_t A_{2,itr}\right)\right]$$
$$-\kappa_2 h^2\left\{H^{(r)\prime}E\left[F_t(\bar{C}_{2t}^{(r)} + \bar{C}_{3t}^{(r)} + \bar{C}_{4t}^{(r)})'\right] + E[\bar{C}_{1t}^{(r)}\bar{C}_{1t}^{(r)\prime}]\right\}H^{(r)-1}\lambda_{ir} + O_P(T^{-1/2} + h^3).$$

Finally, by Assumption A.6(ii), we know $\frac{\sqrt{h}}{\sqrt{T}}\sum_{s=1}^{T} k_{h,sr}F_s\varepsilon_{is} \xrightarrow{d} N(0,\Omega_{i,r})$. It follows that $\sqrt{Th}D_1(i,r) = \frac{1}{\sqrt{Th}}H^{(r)\prime}\sum_{s=1}^{T} K_r^*(\frac{s-r}{Th})F_s\varepsilon_{is} \xrightarrow{d} N(0,\left(Q_r^{-1}\right)'\Omega_{i,r}\left(Q_r^{-1}\right))$. This completes the proof of Theorem 3.3(ii).

(iii) Noting that $Y_{it} - X_{it}'\hat{\beta}_t = -X_{it}'\hat{\delta}_t - [\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it}]'H^{(t)\prime}F_t + \hat{\lambda}_{it}'H^{(t)\prime}F_t + \varepsilon_{it}$, we have

$$\hat{F}_t - H^{(t)'}F_t = \hat{S}_{\lambda,t}^{-1}\frac{1}{N}\sum_{i=1}^{N}\hat{\lambda}_{it}(Y_{it} - X_{it}'\hat{\beta}_t) - H^{(t)'}F_t$$
$$= \hat{S}_{\lambda,t}^{-1}H^{(t)-1}\frac{1}{N}\sum_{i=1}^{N}\lambda_{it}\varepsilon_{it} + \hat{S}_{\lambda,t}^{-1}\frac{1}{N}\sum_{i=1}^{N}(\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it})\varepsilon_{it}$$
$$-\hat{S}_{\lambda,t}^{-1}\frac{1}{N}\sum_{i=1}^{N}\hat{\lambda}_{it}[\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it}]'H^{(t)\prime}F_t - \hat{S}_{\lambda,t}^{-1}\frac{1}{N}\sum_{i=1}^{N}\hat{\lambda}_{it}X_{it}'\hat{\delta}_t \equiv \sum_{\ell=1}^{4}A_\ell(t), \tag{A.9}$$

where $\hat{S}_{\lambda,t} = \frac{1}{N}\sum_{i=1}^{N}\hat{\lambda}_{it}\hat{\lambda}_{it}'$. By Lemma A.5(i)-(ii) in Su and Wang (2020),

$$\hat{S}_{\lambda,t} = \frac{1}{N}\sum_{i=1}^{N}\hat{\lambda}_{it}\hat{\lambda}_{it}' = Q_t\Sigma_{\Lambda_t}Q_t' + o_P(1) \text{ and } \frac{1}{\sqrt{N}}\sum_{i=1}^{N}(\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it})\varepsilon_{it} = o_P(1).$$

Then $\sqrt{N}A_2(t) = o_P(1)$. For $A_4(t)$, we have $\left\|\sqrt{N}A_4(t)\right\| \leq \left\|\frac{1}{N}\sum_{i=1}^{N}\hat{\lambda}_{it}X_{it}'\right\|\sqrt{N}\|\hat{\delta}_t\| = O_P(1)\sqrt{N}O_P(N^{-1} + (Th)^{-1} + h^2)$ by Theorem 3.2. Noting that $NTh^5 = O(1)$ and $1/(Th) = o(1)$ implies that $Nh^4 = o(1)$, $\left\|\sqrt{N}A_4(t)\right\| = o_P(1)$ under Assumption A.4(ii) and the additional condition $NTh^5 = O(1)$.

For $A_1(t)$, we have by Assumption A.6(ii),

$$\sqrt{N}A_1(t) = \hat{S}_{\lambda,t}^{-1}H^{(t)-1}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_{it}\varepsilon_{it} = \left(Q_t\Sigma_{\Lambda_t}Q_t'\right)^{-1}Q_t\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_{it}\varepsilon_{it} + o_P(1)$$
$$\xrightarrow{d} N\left(0,\left(Q_t\Sigma_{\Lambda_t}Q_t'\right)^{-1}Q_t\Gamma_{tt}Q_t'\left(Q_t\Sigma_{\Lambda_t}Q_t'\right)^{-1}\right) = N\left(0,\left(\Sigma_{\Lambda_t}^{-1}Q_t^{-1}\right)'\Gamma_{tt}\Sigma_{\Lambda_t}^{-1}Q_t^{-1}\right).$$

Last, we show that $A_3(t)$ contributes to the asymptotic bias. To this end, we make the following decomposition

$$-\sqrt{N}A_3(t) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}[\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it}][\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it}]'H^{(t)\prime}F_t + \frac{1}{\sqrt{N}}\sum_{i=1}^{N}H^{(t)-1}\lambda_{it}[\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it}]'H^{(t)\prime}F_t$$

$$\equiv A_{3,1}(t) + A_{3,2}(t).$$

Noting that $\left\|\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it}\right\| = O_P((Th)^{-1/2} + h^2)$ by Theorem 3.3(ii), we have:

$$\|A_{3,1}(t)\| \lesssim \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left\|\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it}\right\|^2 = O_P(N^{1/2}((Th)^{-1} + h^4)) = o_P(1).$$

Next, $A_{3,2}(t) = \sum_{l=1}^{11} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} H^{(t)-1}\lambda_{it} D_l(i,r)' H^{(t)\prime} F_t \equiv \sum_{l=1}^{11} A_{3,2l}(t)$ by (A.8). Su and Wang (2020) have studied the terms $A_{3,21}(t)$ to $A_{3,28}(t)$. Following their analysis, we can show that $A_{3,2l}(t) = o_P(1)$ for $l = 1, 2, 3, 4, 6$. Now, we consider the terms $A_{3,25}(t)$, $A_{3,27}(t)$, and $A_{3,28}(t)$. First,

$$\begin{aligned}
A_{3,25}(t) &= H^{(t)-1} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \lambda_{it} \frac{1}{T} \Delta_i^{(t)\prime} F^{(t)} H^{(t)} H^{(t)\prime} F_t \\
&= H^{(t)-1} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \lambda_{it} \left\{ \frac{1}{T} \sum_{s=1}^{T} k_{h,st}^* \left(X_{is}' \beta_t^{(1)} + \lambda_{it}^{(1)\prime} F_s \right) F_s' \frac{s-t}{T} \right\} H^{(t)} H^{(t)\prime} F_t \\
&+ \frac{1}{2\sqrt{N}} \sum_{i=1}^{N} H^{(t)-1} \lambda_{it} \left\{ \frac{1}{T} \sum_{s=1}^{T} k_{h,st}^* \left(X_{is}' \beta_t^{(2)} + \lambda_{it}^{(2)\prime} F_s \right) F_s' (\frac{s-t}{T})^2 \right\} H^{(t)} H^{(t)\prime} F_t + O_P(N^{1/2} h^3) \\
&= \frac{h^2}{2} \sqrt{N} H^{(t)-1} \left\{ \frac{1}{N} \sum_{i=1}^{N} \lambda_{it} \beta_t^{(2)\prime} E(X_{is} F_s') + \frac{\Lambda_t' \Lambda_t^{(2)}}{N} \Sigma_F \right\} \Sigma_F^{-1} \kappa_2 F_t + o_P(1),
\end{aligned}$$

where we use the fact that $\frac{1}{T}\sum_{s=1}^{T} k_{h,st}^* F_s' [X_{is}' \beta_t^{(1)} + \lambda_{it}^{(1)\prime} F_s] \frac{s-t}{T} = O_P((Th)^{-1/2} h) = o_P(N^{-1/2})$ and $H^{(t)} H^{(t)\prime} = \Sigma_F^{-1} + O_P(C_{NT}^{-1})$ by Lemma A.2(iv) and Theorem 3.2. For $A_{3,27}(t)$, we have

$$\begin{aligned}
A_{3,27}(t) &= \frac{1}{\sqrt{N}T} \sum_{i=1}^{N} \sum_{s=1}^{T} H^{(t)-1} \lambda_{it} \lambda_{it}' (H^{(t)-1})' B_s^{(t)} \hat{F}_s^{(t)\prime} H^{(t)\prime} F_t \\
&= \sum_{l=1}^{3} H^{(t)-1} \left(\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \lambda_{it} \lambda_{it}' \right) H^{(t)-1\prime} \left(\frac{1}{T} \sum_{s=1}^{T} B_s^{(t)} \varphi_{l,st}' \right) H^{(t)\prime} F_t \equiv \sum_{l=1}^{3} A_{3,27l}(t).
\end{aligned}$$

For $A_{3,271}(t)$, we have

$$\|A_{3,271}(t)\| \lesssim \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \lambda_{it} \lambda_{it}' \right\| \left\| \frac{1}{T} \sum_{s=1}^{T} B_s^{(t)} \left[\hat{F}_s^{(t)} - H^{(t)\prime} F_s^{(t)} - B_s^{(t)} \right]' \right\| = N^{1/2} O_P((C_{NT}^{-1} + \|\hat{\delta}_r\|) h) = o_P(1).$$

For $A_{3,272}(t)$ and $A_{3,273}(t)$, we use $B_s^{(t)} = C_{1s}^{(t)} \frac{s-t}{T} + C_{2s}^{(t)} (\frac{s-t}{T})^2 + (C_{3s}^{(t)} + C_{4s}^{(t)}) h^2 \kappa_2$ to obtain

$$\begin{aligned}
A_{3,272}(t) &= \frac{1}{\sqrt{N}T} \sum_{i=1}^{N} \sum_{s=1}^{T} H^{(t)-1} \lambda_{it} \lambda_{it}' H^{(t)-1\prime} B_s^{(t)} F_s^{(t)\prime} H^{(t)} H^{(t)\prime} F_t \\
&= \sqrt{N} H^{(t)-1} (\frac{1}{N} \Lambda_t' \Lambda_t) H^{(t)-1\prime} \frac{1}{T} \sum_{s=1}^{T} k_{h,st}^* \left[C_{1s}^{(t)} \frac{s-t}{T} + C_{2s}^{(t)} (\frac{s-t}{T})^2 + (C_{3s}^{(t)} + C_{4s}^{(t)}) h^2 \kappa_2 \right] F_s' H^{(t)} H^{(t)\prime} F_t \\
&= \sqrt{N} h^2 H^{(t)-1} \Sigma_{\Lambda_t} H^{(t)-1\prime} E[(\bar{C}_{2s}^{(t)} + \bar{C}_{3s}^{(t)} + \bar{C}_{4s}^{(t)}) F_s'] \Sigma_F^{-1} \kappa_2 F_t + o_P(1),
\end{aligned}$$

and

$$A_{3,273}(t) = \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{s=1}^{T} H^{(t)-1}\lambda_{it}\lambda_{it}' H^{(t)-1\prime} B_s^{(t)} B_s^{(t)\prime} H^{(t)\prime} F_t$$
$$= \sqrt{N} H^{(t)-1}\left[\frac{1}{N}\sum_{i=1}^{N}\lambda_{it}\lambda_{it}'\right] H^{(t)-1\prime}\left[\frac{1}{T}\sum_{s=1}^{T} k_{h,st}^{*} C_{1s}^{(t)} C_{1s}^{(t)\prime}(\frac{s-t}{T})^2\right] H^{(t)\prime}F_t + O_P(N^{1/2}h^3)$$
$$= \sqrt{N}h^2\kappa_2 H^{(t)-1}\Sigma_{\Lambda_t} H^{(t)-1\prime} E[\bar{C}_{1s}^{(t)}\bar{C}_{1s}^{(t)\prime}] H^{(t)\prime}F_t + o_P(1),$$

where $\bar{C}_{ls}^{(t)}$ is defined analogously to $C_{ls}^{(t)}$ with $V_{NT}^{(t)}$ and $H^{(t)}$ replaced by their probability limits for $l = 1, 2, 3$. It follows that $A_{3,27}(t) = \sqrt{N}h^2\kappa_2 H^{(t)-1}\Sigma_{\Lambda_t} H^{(t)-1\prime}\{E[(\bar{C}_{2s}^{(t)} + \bar{C}_{3s}^{(t)} + \bar{C}_{4s}^{(t)})F_s'] + E(\bar{C}_{1s}^{(t)}\bar{C}_{1s}^{(t)\prime})H^{(t)\prime}\}F_t + o_P(1)$. For $A_{3,28}(t)$, we have

$$A_{3,28}(t) = H^{(t)-1}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_{it}\frac{1}{T}\Delta_i(t)' B^{(t)} H^{(t)\prime}F_t$$
$$= \sqrt{N}H^{(t)-1}\frac{1}{N}\sum_{i=1}^{N}\lambda_{it}\lambda_{it}^{(1)\prime}\frac{1}{T}\sum_{s=1}^{T} k_{h,st}^{*} F_s C_{1s}^{(t)\prime}(\frac{s-t}{T})^2 H^{(t)\prime}F_t + O_P(N^{1/2}h^3)$$
$$= \sqrt{N}h^2\kappa_2 H^{(t)-1}\Sigma_{\Lambda_t} E[F_s\bar{C}_{1s}^{(t)\prime}] H^{(t)\prime}F_t + o_p(1).$$

In addition, it is easy to see that $\sqrt{N}\sum_{l=8}^{11} A_{3,2l}(t) = \sqrt{N}O_P(\|\hat{\delta}_t\|) = o_P(1)$. In sum, we have

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\hat{\lambda}_{it}\left[\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it}\right]' H^{(t)\prime}F_t = \frac{h^2}{2}\sqrt{N}H^{(t)-1}\left\{\left[\frac{1}{N}\sum_{i=1}^{N}\lambda_{it}\beta_r^{(2)\prime}E(X_{is}F_s')\right] + \frac{\Lambda_t'\Lambda_t^{(2)}}{N}\Sigma_F\right\}\Sigma_F^{-1}\kappa_2 F_t$$
$$- \sqrt{N}h^2\kappa_2 H^{(t)-1}\Sigma_{\Lambda_t} H^{(t)-1\prime}\left\{E[(\bar{C}_{2s}^{(t)} + \bar{C}_{3s}^{(t)} + \bar{C}_{4s}^{(t)})F_s'] + E(\bar{C}_{1s}^{(t)}\bar{C}_{1s}^{(t)\prime})H^{(t)\prime}\right\}F_t$$
$$+ \sqrt{N}h^2 H^{(t)-1}\Sigma_{\Lambda_t} E(F_s\bar{C}_{1s}^{(t)\prime})H^{(t)\prime}F_t\kappa_2 + o_P(1)$$
$$\equiv \sqrt{N}\tilde{B}_F(t)\kappa h^2 F_t + o_P(1),$$

where

$$\tilde{B}_F(t) = H^{(t)-1}\Lambda_t' E\left(A_{2,st}F_s'\right)\Sigma_F^{-1} - H^{(t)-1}\Sigma_{\Lambda_t} H^{(t)-1\prime}\{E[(\bar{C}_{2s}^{(t)} + \bar{C}_{3s}^{(t)} + \bar{C}_{4s}^{(t)})F_s'] + E(\bar{C}_{1s}^{(t)}\bar{C}_{1s}^{(t)\prime})H^{(t)\prime}\}$$
$$+ H^{(t)-1}\Sigma_{\Lambda_t} E(F_s\bar{C}_{1s}^{(t)\prime})H^{(t)\prime}.$$

Therefore, we have

$$\sqrt{N}A_3(t) = \hat{S}_{\lambda,t}^{-1}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\hat{\lambda}_{it}\left[\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it}\right]' H^{(t)\prime}F_t = \sqrt{N}\left(Q_t\Sigma_{\Lambda_t}Q_t'\right)^{-1}\tilde{B}_F(t)\kappa h^2 F_t + o_P(\sqrt{N}h^3)$$
$$= \sqrt{N}B_F(t)F_t + o_P(1),$$

where $B_F(t) = \left(Q_t\Sigma_{\Lambda_t}Q_t'\right)^{-1}\tilde{B}_F(t)\kappa h^2$. Under the additional condition that $Nh^4 = o(1)$, $\sqrt{N}A_3(t) = o_P(1)$.

Combining these results, we have $\sqrt{N}[\hat{F}_t - H^{(t)\prime}F_t] \xrightarrow{d} N(0, \left(\Sigma_{\Lambda_t}^{-1}Q_t^{-1}\right)'\Gamma_{tt}\Sigma_{\Lambda_t}^{-1}Q_t^{-1})$.

(iv) Noting that $C_{it}^0 = \lambda_{it}'F_t$ and $\hat{C}_{it} = \hat{\lambda}_{it}'\hat{F}_t$, we have

$$\hat{C}_{it} - C_{it}^0$$
$$= \hat{\lambda}_{it}'\hat{F}_t - \lambda_{it}'F_t$$
$$= [\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it} - B_\Lambda(i,t)]'[\hat{F}_t - H^{(t)\prime}F_t - B_F(t)] + B_\Lambda(i,t)'B_F(t) + [\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it} - B_\Lambda(i,t)]'H^{(t)\prime}F_t$$

$$
\begin{aligned}
&+ [\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it} - B_\Lambda(i,t)]' B_F(t) + \lambda_{it}'(H^{(t)-1})'[\hat{F}_t - H^{(t)\prime}F_t - B_F(t)] + \lambda_{it}'(H^{(t)-1})'B_F(t) \\
&+ B_\Lambda(i,t)'[\hat{F}_t - H^{(t)\prime}F_t - B_F(t)] + B_\Lambda(i,t)'H^{(t)\prime}F_t \\
&\equiv \sum_{l=1}^{8} C_{it,l}. \qquad \text{(A.10)}
\end{aligned}
$$

By Theorems 3.3(i)-(ii), $C_{it,1} = O_P((NTh)^{-1/2})$, $C_{it,2} = O_P(h^4)$, $C_{it,4} = O_P(T^{-1/2}h^{-1/2}h^2)$, $C_{it,7} = O_P(N^{-1/2}h^2)$. By the proof of Theorem 3.2(ii), $\sqrt{Th}(\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it} - B_\Lambda(i,t)) = \frac{\sqrt{h}}{\sqrt{T}} H^{(t)\prime} \sum_{s=1}^{T} k_{h,st}^* F_s \varepsilon_{is} + o_P(1)$. By the proof of Theorem 3.3(iii), $\sqrt{N}(\hat{F}_t - H^{(t)'}F_t - B_F(t)) = (Q_t \Sigma_{\Lambda_t} Q_t')^{-1} Q_t \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \lambda_{it}\varepsilon_{it} + o_P(1)$. By Lemma A.2(iii)-(iv) and Theorem 3.2, $H^{(t)} = Q_t^{-1} + O_P(C_{NT}^{-1})$ and $H^{(t)}H^{(t)\prime} = \Sigma_F^{-1} + O_P(C_{NT}^{-1})$. Define $\bar{C}_{NT} = \min\{\sqrt{Th}, \sqrt{N}\}$. It follows that

$$
\begin{aligned}
C_{it,6} + C_{it,8} &= \lambda_{it}'(H^{(t)-1})'B_F(t) + B_\Lambda(i,t)'H^{(t)\prime}F_t \\
&= \lambda_{it}'Q_t'B_F(t) + B_\Lambda(i,t)'(Q_t^{(-1)})'F_t + o_P(\bar{C}_{NT}^{-1}) \equiv B_C(i,t) + o_P(\bar{C}_{NT}^{-1}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\bar{C}_{NT}[\hat{C}_{it} - C_{it}^0 - B_C(i,t)] \\
&= \frac{\bar{C}_{NT}}{\sqrt{N}} \lambda_{it}' Q_t' (Q_t \Sigma_{\Lambda_t} Q_t')^{-1} Q_t \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \lambda_{it}\varepsilon_{it} + \frac{\bar{C}_{NT}}{\sqrt{Th}} F_t' H^{(t)} H^{(t)\prime} \frac{\sqrt{h}}{\sqrt{T}} \sum_{s=1}^{T} k_{h,st}^* F_s \varepsilon_{is} + o_P(1) \\
&= \frac{\bar{C}_{NT}}{\sqrt{N}} \psi_{1it} + \frac{\bar{C}_{NT}}{\sqrt{Th}} \psi_{2it} + o_P(1),
\end{aligned}
$$

where $\psi_{1it} \equiv \lambda_{it}' \Sigma_{\Lambda_t}^{-1} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \lambda_{it}\varepsilon_{it}$ and $\psi_{2it} \equiv F_t' \Sigma_F^{-1} \frac{\sqrt{h}}{\sqrt{T}} \sum_{s=1}^{T} k_{h,st}^* F_s \varepsilon_{is}$. By Assumption A.2, $\psi_{1it} \xrightarrow{d} N(0, V_{1it})$ and $\psi_{2it} \xrightarrow{d} N(0, V_{2it})$. It is easy to show that ξ_{1it} and ξ_{2it} are asymptotically independent. Consequently, we have $V_{it}^{-1/2}\bar{C}_{NT}(\hat{C}_{it} - C_{it}^0 - B_C(i,t)) \xrightarrow{d} N(0,1)$, where $V_{it} = \frac{\bar{C}_{NT}^2}{N} V_{1it} + \frac{\bar{C}_{NT}^2}{Th} V_{2it}$. ■

Proof of Theorem 3.4. By (A.5) in the proof of Theorem 3.2 and the fact that $\hat{\delta}_t \equiv \hat{\beta}_t - \beta_t = O_P(N^{-1} + (Th)^{-1} + h^2)$ and

$$
\begin{aligned}
&D^{(t)}(\hat{F}^{(t)})\sqrt{NTh}(\hat{\beta}_t - \beta_t) \\
&= \frac{\sqrt{NTh}}{NT} \sum_{i=1}^{N} \left[X_i^{(t)\prime} - \frac{1}{N} \sum_{j=1}^{N} a_{ij}^{(t)} X_j^{(t)\prime} \right] M_{\hat{F}^{(t)}} \varepsilon_i^{(t)} + \frac{\sqrt{NTh}}{NT} \sum_{i=1}^{N} \left[X_i^{(t)\prime} - \frac{1}{N} \sum_{j=1}^{N} a_{ij}^{(t)} X_j^{(t)\prime} \right] M_{\hat{F}^{(t)}} \Delta_i^{(t)} \\
&- \frac{\sqrt{NTh}}{N^2T^2} \sum_{i=1}^{N} \sum_{j=1}^{N} X_i^{(t)\prime} M_{\hat{F}^{(t)}} \varepsilon_j^{(t)} \varepsilon_j^{(t)\prime} \hat{F}^{(t)} G^{(t)} \lambda_{it} + o_P(1),
\end{aligned}
$$

where we use the condition $NTh^7 = o(1)$ in Assumption A4(ii) to ensure that $\sqrt{NTh}O_P(C_{NT}^{-1} + \|\hat{\delta}_t\|^{1/2} + h)O_P(\|\hat{\delta}_t\|) = o_P(1)$. Let $\tilde{Z}_i^{(t)} = M_{\check{F}^{(t)}}[X_i^{(t)} - \frac{1}{N}\sum_{j=1}^{N} a_{ij}^{(t)} X_j^{(t)}]$. Then

$$
\begin{aligned}
&\hat{D}^{(t)}(\check{F}^{(t)})(\hat{\beta}_t^{bc} - \beta_t) \\
&= \left[\hat{D}^{(t)}(\check{F}^{(t)}) - D^{(t)}(\hat{F}^{(t)})\right] \left[\hat{\beta}_t - \beta_t\right] + D^{(t)}(\hat{F}^{(t)}) \left[\hat{\beta}_t - \beta_t\right] - \left[\hat{B}_{1\beta}^{(t)} + \frac{1}{Th}\hat{B}_{2\beta}^{(t)} + \frac{1}{N}\hat{B}_{3\beta}^{(t)} + \frac{1}{Th}\hat{B}_{4\beta}^{(t)}\right] \\
&= \left[\hat{D}^{(t)}(\check{F}^{(t)}) - D^{(t)}(\hat{F}^{(t)})\right] \left[\hat{\beta}_t - \beta_t\right] + \left[\frac{1}{NT} \sum_{i=1}^{N} \tilde{Z}_i^{(t)\prime} \varepsilon_i^{(t)} - \frac{1}{Th}\hat{B}_{2\beta}^{(t)} - \frac{1}{N}\hat{B}_{3\beta}^{(t)}\right] + \frac{1}{NT} \sum_{i=1}^{N} \tilde{Z}_i^{(t)\prime} \Delta_i^{(t)}
\end{aligned}
$$

$$
- \left[\frac{1}{N^2T^2} \sum_{i=1}^{N} \sum_{j=1}^{N} X_i^{(t)\prime} M_{\hat{F}^{(t)}} \varepsilon_j^{(t)} \varepsilon_j^{(t)\prime} \hat{F}^{(t)} G^{(t)} \lambda_{it} - \frac{1}{Th} \hat{B}_{4\beta}^{(t)} \right] + o_P((NTh)^{-1/2})
$$

$$
\equiv Q_1^{(t)} + Q_2^{(t)} + Q_3^{(t)} + Q_4^{(t)} + O_P(h^4) + o_P((NTh)^{-1/2}).
$$

As in the proof of Lemma A.9, we can show that

$$
\left\| \hat{D}^{(t)}(\check{F}^{(t)}) - D^{(t)}(\hat{F}^{(t)}) \right\| \leq \left\| \hat{D}^{(t)}(\check{F}^{(t)}) - D^{(t)}(\check{F}^{(t)}) \right\| + \left\| D^{(t)}(\check{F}^{(t)}) - D^{(t)}(\hat{F}^{(t)}) \right\| = O_P(C_{NT}^{-2} + \|\hat{\delta}_t\|^{1/2}).
$$

As a result, $Q_1^{(t)} = O_P(C_{NT}^{-2} + \|\hat{\delta}_t\|^{1/2})\|\hat{\delta}_t\| = o_P((NTh)^{-1/2})$. This rate can be strengthened to be uniform in t under Assumption A.7. By Lemmas A.6 and A.8, we have

$$
Q_2^{(t)} = \frac{1}{NT} \sum_{i=1}^{N} \sum_{s=1}^{T} \mathfrak{X}_{is}^{(t)} \varepsilon_{is}^{(t)} + \frac{1}{N} \left[\xi_{NT}^{(t)} - \hat{B}_{3\beta}^{(t)} \right] + \frac{1}{Th} \left[B_{2\beta}^{(t)} - \hat{B}_{2\beta}^{(t)} \right] \equiv Q_{2,1}^{(t)} + Q_{2,2}^{(t)} + Q_{2,3}^{(t)}.
$$

Note that $Q_{2,1}^{(t)}$ contributes to the asymptotic variance of $\hat{\beta}_t - \beta_t$: $\sqrt{NTh} Q_{2,1}^{(t)} \xrightarrow{d} N(0, \Omega_t)$. By Assumption A.5, $\max_t \left\| \frac{1}{NT} \sum_{i=1}^{N} \sum_{s=1}^{T} \mathfrak{X}_{is}^{(t)} \varepsilon_{is}^{(t)} \right\| = O_P((NTh)^{-1/2} (\ln T)^{1/2})$. For notational simplicity, let $\hat{\varepsilon}^{(t)}(i,j) = \frac{1}{T} \sum_{s=1}^{T} k_{h,st}^{*} \hat{\varepsilon}_{is} \hat{\varepsilon}_{js}$, and $\varepsilon^{(t)}(i,j) = \frac{1}{T} \sum_{s=1}^{T} k_{h,st}^{*} \varepsilon_{is} \varepsilon_{js}$. Then

$$
\begin{aligned}
Q_{2,2}^{(t)} &= \frac{1}{N^2} \sum_{i,j=1}^{N} \frac{1}{T} [X_i^{(t)} - \hat{V}_i^{(t)}]' \check{F}^{(t)} (\frac{1}{N} \hat{\Lambda}_t' \hat{\Lambda}_t)^{-1} \hat{\lambda}_{jt} \hat{\varepsilon}^{(t)}(i,j) \\
&\quad - \frac{1}{N^2} \sum_{i,j=1}^{N} \frac{1}{T} [X_i^{(t)} - V_i^{(t)}]' F^{(t)} [\frac{1}{T} F^{(t)\prime} F^{(t)}]^{-1} (\frac{1}{N} \Lambda_t' \Lambda_t)^{-1} \lambda_{jt} \varepsilon^{(t)}(i,j) \\
&= \frac{1}{N^2} \sum_{i,j=1}^{N} \left[\frac{1}{T} X_i^{(t)\prime} \check{F}^{(t)} (\frac{1}{N} \hat{\Lambda}_t' \hat{\Lambda}_t)^{-1} \hat{\lambda}_{jt} \hat{\varepsilon}^{(t)}(i,j) - \frac{1}{T} X_i^{(t)\prime} F^{(t)} [\frac{1}{T} F^{(t)\prime} F^{(t)}]^{-1} (\frac{1}{N} \Lambda_t' \Lambda_t)^{-1} \lambda_{jt} \varepsilon^{(t)}(i,j) \right] \\
&\quad - \frac{1}{N^2} \sum_{i,j=1}^{N} \left[\frac{1}{T} \hat{V}_i^{(t)\prime} \check{F}^{(t)} (\frac{1}{N} \hat{\Lambda}_t' \hat{\Lambda}_t)^{-1} \hat{\lambda}_{jt} \hat{\varepsilon}^{(t)}(i,j) - \frac{1}{T} V_i^{(t)\prime} F^{(t)} [\frac{1}{T} F^{(t)\prime} F^{(t)}]^{-1} (\frac{1}{N} \Lambda_t' \Lambda_t)^{-1} \lambda_{jt} \varepsilon^{(t)}(i,j) \right] \\
&\equiv Q_{2,21}^{(t)} + Q_{2,22}^{(t)}
\end{aligned}
$$

We first consider the term $Q_{2,21}^{(t)}$. Using the identity $\hat{a}\hat{b}\hat{c}\hat{d} - abcd = (\hat{a} - a)\hat{b}\hat{c}\hat{d} + a(\hat{b} - b)\hat{c}\hat{d} + ab(\hat{c} - c)\hat{d} + abc(\hat{d} - d)$, we can decompose $Q_{2,21}^{(t)}$ into four terms:

$$
\begin{aligned}
Q_{2,21}^{(t)} &= \frac{1}{N^2} \sum_{i,j=1}^{N} \frac{1}{T} X_i^{(t)\prime} [\check{F}^{(t)} - F^{(t)} H^{(t)}] (\frac{1}{N} \hat{\Lambda}_t' \hat{\Lambda}_t)^{-1} \hat{\lambda}_{jt} \hat{\varepsilon}^{(t)}(i,j) \\
&\quad + \frac{1}{N^2} \sum_{i,j=1}^{N} \frac{1}{T} X_i^{(t)\prime} F^{(t)} \left[H^{(t)} (\frac{1}{N} \hat{\Lambda}_t' \hat{\Lambda}_t)^{-1} - [\frac{1}{T} F^{(t)\prime} F^{(t)}]^{-1} (\frac{1}{N} \Lambda_t' \Lambda_t)^{-1} H^{(t)} \right] \hat{\lambda}_{jt} \hat{\varepsilon}^{(t)}(i,j) \\
&\quad + \frac{1}{N^2} \sum_{i,j=1}^{N} \frac{1}{T} X_i^{(t)\prime} F^{(t)} [\frac{1}{T} F^{(t)\prime} F^{(t)}]^{-1} (\frac{1}{N} \Lambda_t' \Lambda_t)^{-1} H^{(t)} \left[\hat{\lambda}_{jt} - H^{(t)-1} \lambda_{jt} \right] \hat{\varepsilon}^{(t)}(i,j) \\
&\quad + \frac{1}{N^2} \sum_{i,j=1}^{N} \frac{1}{T} X_i^{(t)\prime} F^{(t)} [\frac{1}{T} F^{(t)\prime} F^{(t)}]^{-1} (\frac{1}{N} \Lambda_t' \Lambda_t)^{-1} \lambda_{jt} \left[\hat{\varepsilon}^{(t)}(i,j) - \varepsilon^{(t)}(i,j) \right] \equiv \sum_{l=1}^{4} Q_{2,21l}^{(t)}.
\end{aligned}
$$

For $Q^{(t)}_{2,211}$, we have

$$\begin{aligned}\left\|Q^{(t)}_{2,211}\right\| &\leq \max_t \left\|\frac{1}{N^2}\sum_{i,j=1}^{N}\frac{1}{T}X_i^{(t)\prime}[\check{F}^{(t)}-F^{(t)}H^{(t)}](\frac{1}{N}\hat{\Lambda}_t^{\prime}\hat{\Lambda}_t)^{-1}\hat{\lambda}_{jt}\hat{\varepsilon}^{(t)}(i,j)\right\| \\ &\lesssim \max_{i,t}\left\|\frac{1}{T}X_i^{(t)\prime}[\check{F}^{(t)}-F^{(t)}\tilde{H}^{(t)}]\right\| \max_t \frac{1}{N^2}\sum_{i,j=1}^{N}\left\|\hat{\lambda}_{jt}\hat{\varepsilon}^{(t)}(i,j)\right\|.\end{aligned}$$

Following the proof of Theorem 3.3(iii) and using Assumption A.7, we can show that $\frac{1}{T}X_i^{(t)\prime}[\check{F}^{(t)}-F^{(t)}H^{(t)}]=O_P\left(N^{-1}+(Th)^{-1}+h^2\right)$ for each (i,t) and $\max_{i,t}\left\|\frac{1}{T}X_i^{(t)\prime}[\check{F}^{(t)}-F^{(t)}H^{(t)}]\right\|=O_P\left((N^{-1}+(Th)^{-1})\ln T+h^2\right)$. Next, note that

$$\begin{aligned}&\frac{1}{N^2}\sum_{i=1}^{N}\left\|\sum_{j=1}^{N}\hat{\lambda}_{jt}\hat{\varepsilon}^{(t)}(i,j)\right\| \\ \leq\ & \frac{1}{N^2}\sum_{i=1}^{N}\left\|\sum_{j=1}^{N}[\hat{\lambda}_{jt}-H^{(t)-1}\lambda_{jt}]\hat{\varepsilon}^{(t)}(i,j)\right\|+\left\|H^{(t)-1}\right\|\frac{1}{N^2}\sum_{i=1}^{N}\left\|\sum_{j=1}^{N}\lambda_{jt}\hat{\varepsilon}^{(t)}(i,j)\right\| \\ \lesssim\ & \frac{1}{N^2}\sum_{i=1}^{N}\left\|\sum_{j=1}^{N}[\hat{\lambda}_{jt}-H^{(t)-1}\lambda_{jt}]\varepsilon^{(t)}(i,j)\right\|+\frac{1}{N^2}\sum_{i=1}^{N}\left\|\sum_{j=1}^{N}[\hat{\lambda}_{jt}-H^{(t)-1}\lambda_{jt}][\hat{\varepsilon}^{(t)}(i,j)-\varepsilon^{(t)}(i,j)]\right\| \\ &+\frac{1}{N^2}\sum_{i=1}^{N}\left\|\sum_{j=1}^{N}\lambda_{jt}\varepsilon^{(t)}(i,j)\right\|+\frac{1}{N^2}\sum_{i=1}^{N}\left\|\sum_{j=1}^{N}\lambda_{jt}[\hat{\varepsilon}^{(t)}(i,j)-\varepsilon^{(t)}(i,j)]\right\|\equiv\sum_{l=1}^{4}a_l^{(t)}.\end{aligned}$$

For $a_1^{(t)}$, it suffices to consider the rough probability bound:

$$\begin{aligned}\max_t a_1^{(t)} &= \max_t \frac{1}{N^2}\sum_{i=1}^{N}\left\|\sum_{j=1}^{N}[\hat{\lambda}_{jt}-H^{(t)-1}\lambda_{jt}]\varepsilon^{(t)}(i,j)\right\| \\ &\leq \max_t\left\{\frac{1}{N}\sum_{j=1}^{N}\left\|\hat{\lambda}_{jt}-H^{(t)-1}\lambda_{jt}\right\|^2\right\}^{1/2}\left\{\frac{1}{N^2}\sum_{i,j=1}^{N}\left\|\frac{1}{T}\sum_{s=1}^{T}k^*_{h,st}\varepsilon_{is}\varepsilon_{js}\right\|^2\right\}^{1/2} \\ &= O_P((Th/\ln T)^{-1/2}+h^2)O_P(N^{-1/2}+(Th/\ln T)^{-1/2}),\end{aligned}$$

where we use Theorem 3.5(i) below and the fact that

$$\begin{aligned}&\max_t \frac{1}{N^2}\sum_{i,j=1}^{N}\left|\frac{1}{T}\sum_{s=1}^{T}k^*_{h,st}\varepsilon_{is}\varepsilon_{js}\right|^2 \\ \leq\ & \max_t \frac{2}{N^2}\sum_{i,j=1}^{N}\left|\frac{1}{T}\sum_{s=1}^{T}k^*_{h,st}E\left(\varepsilon_{is}\varepsilon_{js}\right)\right|^2+\max_t\frac{2}{N^2}\sum_{i,j=1}^{N}\left|\frac{1}{T}\sum_{s=1}^{T}k^*_{h,st}[\varepsilon_{is}\varepsilon_{js}-E(\varepsilon_{is}\varepsilon_{js})]\right|^2 \\ \leq\ & \max_t\frac{2}{T^2}\sum_{s=1}^{T}k^*_{h,st}\sum_{r=1}^{T}k^*_{h,rt}\frac{1}{N^2}\sum_{i,j=1}^{N}E\left(\varepsilon_{is}\varepsilon_{js}\right)E\left(\varepsilon_{ir}\varepsilon_{jr}\right)+\max_t\frac{2}{N^2}\sum_{i,j=1}^{N}\left|\frac{1}{T}\sum_{s=1}^{T}k^*_{h,st}[\varepsilon_{is}\varepsilon_{js}-E(\varepsilon_{is}\varepsilon_{js})]\right|^2 \\ \lesssim\ & N^{-1}\max_s\frac{1}{N}\sum_{i,j=1}^{N}\left|E\left(\varepsilon_{is}\varepsilon_{js}\right)\right|+O_P\left((Th/\ln T)^{-1}\right)=O_P\left(N^{-1}+(Th/\ln T)^{-1}\right).\end{aligned}$$

It is easy to show that

$$
\begin{aligned}
& \max_t a_3^{(t)} \\
\lesssim\ & \max_t \frac{1}{N^2} \sum_{i=1}^{N} \left\| \sum_{j=1}^{N} \lambda_{jt} \frac{1}{T} \sum_{s=1}^{T} k_{h,st}^{*} \varepsilon_{is} \varepsilon_{js} \right\| \\
\leq\ & \max_t \frac{1}{N^2} \sum_{i=1}^{N} \left\| \sum_{j=1}^{N} \lambda_{jt} \frac{1}{T} \sum_{s=1}^{T} k_{h,st}^{*} E\left(\varepsilon_{is} \varepsilon_{js}\right) \right\| + \max_t \frac{1}{N^2} \sum_{i=1}^{N} \left\| \frac{1}{NT} \sum_{j=1}^{N} \sum_{s=1}^{T} \lambda_{jt} k_{h,st}^{*} [\varepsilon_{is} \varepsilon_{js} - E\left(\varepsilon_{is} \varepsilon_{js}\right)] \right\| \\
\lesssim\ & \frac{1}{N} \max_{i,s} \frac{1}{N} \sum_{i,j=1}^{N} \left| E\left(\varepsilon_{is} \varepsilon_{js}\right) \right| + O_P((NTh/\ln T)^{-1/2}) = O_P(N^{-1} + (NTh/\ln T)^{-1/2}).
\end{aligned}
$$

Similarly, we can show that $\max_t a_2^{(t)}$ and $\max_t a_4^{(t)}$ are of order no larger than $\max_t a_1^{(t)}$ and $\max_t a_3^{(t)}$, respectively. Then

$$
\max_t \sum_{l=1}^{4} a_l^{(t)} = O_P(Th/\ln T + N^{-1/2} h^2 + (Th/\ln T)^{-1/2} h^2 + N^{-1} + (NTh/\ln T)^{-1/2})
$$

and $\max_t \left\| Q_{2,211}^{(t)} \right\| = O_P\left((N^{-1} + (Th)^{-1}) \ln T + h^2\right) O_P(Th/\ln T + N^{-1/2} h^2 + (Th/\ln T)^{-1/2} h^2 + N^{-1} + (NTh/\ln T)^{-1/2}) = o_P((NTh)^{-1/2})$.

Noting that $H^{(t)} \frac{1}{N} \hat{\Lambda}_t' \hat{\Lambda}_t H^{(t)\prime} - \frac{1}{N} \Lambda_t' \Lambda_t = H^{(t)} (\frac{1}{N} \hat{\Lambda}_t' \hat{\Lambda}_t - \frac{1}{N} \Lambda_t' \Lambda_t) H^{(t)\prime} = H^{(t)} \frac{1}{N} (\hat{\Lambda}_t - \Lambda_t)' (\hat{\Lambda}_t - \Lambda_t) H^{(t)\prime}$ $+ H^{(t)} \frac{1}{N} (\hat{\Lambda}_t - \Lambda_t)' \Lambda_t H^{(t)\prime} + H^{(t)} \frac{1}{N} \Lambda_t' (\hat{\Lambda}_t - \Lambda_t) H^{(t)\prime}$, it is easy to follow the proof of Theorem 3.3(ii) and using the uniform consistency of $H^{(t)}$ to show that $\max_t \left\| H^{(t)} \frac{1}{N} \hat{\Lambda}_t' \hat{\Lambda}_t H^{(t)\prime} - \frac{1}{N} \Lambda_t' \Lambda_t \right\| = O_P\left((Th)^{-1} + N^{-1} \ln T + h^2\right)$. This result, in conjunction with the fact that $\max_t \left\| \left(H^{(t)} H^{(t)\prime}\right)^{-1} - \frac{1}{T} F^{(t)\prime} F^{(t)} \right\| = O_P\left((Th)^{-1} + N^{-1} \ln T + h^2\right)$ established in the proof of Lemma A.11 and the uniform consistency of $H^{(t)}$ and $\frac{1}{T} F^{(t)\prime} F^{(t)}$, implies that

$$
\begin{aligned}
& \left\| H^{(t)} (\frac{1}{N} \hat{\Lambda}_t' \hat{\Lambda}_t)^{-1} - [\frac{1}{T} F^{(t)\prime} F^{(t)}]^{-1} (\frac{1}{N} \Lambda_t' \Lambda_t)^{-1} H^{(t)} \right\| \\
=\ & \left\| \left[H^{(t)} H^{(t)\prime} (H^{(t)} \frac{1}{N} \hat{\Lambda}_t' \hat{\Lambda}_t H^{(t)\prime})^{-1} - [\frac{1}{T} F^{(t)\prime} F^{(t)}]^{-1} (\frac{1}{N} \Lambda_t' \Lambda_t)^{-1} \right] H^{(t)} \right\| \\
\leq\ & \left\| H^{(t)} H^{(t)\prime} - [\frac{1}{T} F^{(t)\prime} F^{(t)}]^{-1} \right\| \left\| (H^{(t)} \frac{1}{N} \hat{\Lambda}_t' \hat{\Lambda}_t H^{(t)\prime})^{-1} \right\| \left\| H^{(t)} \right\| \\
& + \left\| [\frac{1}{T} F^{(t)\prime} F^{(t)}]^{-1} \right\| \left\| (H^{(t)} \frac{1}{N} \hat{\Lambda}_t' \hat{\Lambda}_t H^{(t)\prime})^{-1} - (\frac{1}{N} \Lambda_t' \Lambda_t)^{-1} \right\| \left\| H^{(t)} \right\| = O_P\left((Th)^{-1} + N^{-1} \ln T + h^2\right).
\end{aligned}
$$

Then as in the analysis of $Q_{2,211}^{(t)}$, we can readily show that

$$
\begin{aligned}
& \max_t \left\| Q_{2,212}^{(t)} \right\| \\
\lesssim\ & \max_t \left\| H^{(t)} (\frac{1}{N} \hat{\Lambda}_t' \hat{\Lambda}_t)^{-1} - [\frac{1}{T} F^{(t)\prime} F^{(t)}]^{-1} (\frac{1}{N} \Lambda_t' \Lambda_t)^{-1} H^{(t)} \right\| \frac{1}{N^2} \sum_{i=1}^{N} \left| \sum_{j=1}^{N} \hat{\lambda}_{jt} \hat{\varepsilon}^{(t)}(i,j) \right| \\
=\ & O_P\left((Th)^{-1} + N^{-1} \ln T + h^2\right) O_P(Th/\ln T + N^{-1/2} h^2 + (Th/\ln T)^{-1/2} h^2 + N^{-1} + (NTh/\ln T)^{-1/2}) \\
=\ & o_P((NTh)^{-1/2}).
\end{aligned}
$$

Now, we study $Q_{2,213}^{(t)}$. Note that

$$\begin{aligned} Q_{2,213}^{(t)} &= \frac{1}{N}\sum_{i=1}^{N}\frac{1}{T}X_i^{(t)\prime}F^{(t)}[\frac{1}{T}F^{(t)\prime}F^{(t)}]^{-1}(\frac{1}{N}\Lambda_t^{\prime}\Lambda_t)^{-1}H^{(t)}\frac{1}{N}\sum_{j=1}^{N}\left[\hat{\lambda}_{jt}-H^{(t)-1}\lambda_{jt}\right]\varepsilon^{(t)}(i,j) \\ &\quad+\frac{1}{N^2}\sum_{i,j=1}^{N}\frac{1}{T}X_i^{(t)\prime}F^{(t)}[\frac{1}{T}F^{(t)\prime}F^{(t)}]^{-1}(\frac{1}{N}\Lambda_t^{\prime}\Lambda_t)^{-1}H^{(t)}\left[\hat{\lambda}_{jt}-H^{(t)-1}\lambda_{jt}\right][\hat{\varepsilon}^{(t)}(i,j)-\varepsilon^{(t)}(i,j)] \\ &\equiv Q_{2,213a}^{(t)}+Q_{2,213b}^{(t)}. \end{aligned}$$

Following the proof of Theorem 3.3(ii) and using the fact that $\max_{i\neq j}\max_t\left|\frac{1}{T}\sum_{s=1}^{T}k_{h,st}^{*}\varepsilon_{is}\varepsilon_{js}\right|=O_P((Th/\ln T)^{-1/2})$, we can readily show that uniformly in t and i,

$$\begin{aligned} &\frac{1}{N}\sum_{j=1}^{N}\left[\hat{\lambda}_{jt}-H^{(t)-1}\lambda_{jt}\right]\varepsilon^{(t)}(i,j) \\ =&\ \frac{1}{N}\sum_{j=1}^{N}D_1(j,t)\frac{1}{T}\sum_{s=1}^{T}k_{h,st}^{*}\varepsilon_{is}\varepsilon_{js}+o_P((NTh)^{-1/2}) \\ =&\ \frac{1}{NT^2}H^{(t)\prime}\sum_{j=1}^{N}\sum_{r=1}^{T}k_{h,rt}^{*}F_r\varepsilon_{jr}\sum_{s=1}^{T}k_{h,st}^{*}\varepsilon_{is}\varepsilon_{js}+o_P((NTh)^{-1/2})=H^{(t)\prime}b_i^{(t)}+o_P((NTh)^{-1/2}), \end{aligned}$$

where $b_i^{(t)}=\frac{1}{NT^2}\sum_{j=1}^{N}\sum_{r=1}^{T}\sum_{s=1}^{T}k_{h,st}^{*}k_{h,rt}^{*}F_r\varepsilon_{jr}\varepsilon_{is}\varepsilon_{js}$. Note that

$$\begin{aligned} \max_{i,t}\left\|b_i^{(t)}\right\| &\leq \max_{i,t}\frac{1}{NT^2}\sum_{j=1}^{N}\sum_{r=1}^{T}\sum_{s=1}^{T}k_{h,rt}^{*}k_{h,st}^{*}\left|E(F_r\varepsilon_{jr}\varepsilon_{is}\varepsilon_{js})\right| \\ &\quad+\max_{i,t}\left\|\frac{1}{NT^2}\sum_{j=1}^{N}\sum_{r=1}^{T}\sum_{s=1}^{T}k_{h,rt}^{*}k_{h,st}^{*}[F_r\varepsilon_{jr}\varepsilon_{is}\varepsilon_{js}-E(F_r\varepsilon_{jr}\varepsilon_{is}\varepsilon_{js})]\right\| \\ &\lesssim N^{-1}T^{-1}h^{-2}\max_i\frac{1}{T}\sum_{j=1}^{N}\sum_{r=1}^{T}\sum_{s=1}^{T}\left|E(F_r\varepsilon_{jr}\varepsilon_{is}\varepsilon_{js})\right| \\ &\quad+\max_{i,t}\left\|\frac{1}{NT^2}\sum_{j=1}^{N}\sum_{r=1}^{T}\sum_{s=1}^{T}k_{h,rt}^{*}k_{h,st}^{*}[F_r\varepsilon_{jr}\varepsilon_{is}\varepsilon_{js}-E(F_r\varepsilon_{jr}\varepsilon_{is}\varepsilon_{js})]\right\| \\ &= O(N^{-1}T^{-1}h^{-2})+O_P(N^{-1/2}T^{-1}h^{-1}(\ln T)^{1/2})=o_P((NTh)^{-1/2}). \end{aligned}$$

Then $\max_{i,t}\frac{1}{N}\sum_{j=1}^{N}\left[\hat{\lambda}_{jt}-H^{(t)-1}\lambda_{jt}\right]\varepsilon^{(t)}(i,j)=\frac{1}{N}\sum_{j=1}^{N}\left[\hat{\lambda}_{jt}-H^{(t)-1}\lambda_{jt}\right]\varepsilon^{(t)}(i,j)$. With this, we can readily show that $\max_t\left\|Q_{2,213a}^{(t)}\right\|=o_P((NTh)^{-1/2})$. Noting that

$$\begin{aligned} \hat{\varepsilon}^{(t)}(i,j)-\varepsilon^{(t)}(i,j) &= \frac{1}{T}\sum_{s=1}^{T}[\hat{\varepsilon}_{is}^{(t)}\hat{\varepsilon}_{js}^{(t)}-\varepsilon_{is}^{(t)}\varepsilon_{js}^{(t)}] \\ &= \frac{1}{T}\sum_{s=1}^{T}\{[\hat{\varepsilon}_{is}^{(t)}-\varepsilon_{is}^{(t)}][\hat{\varepsilon}_{js}^{(t)}-\varepsilon_{js}^{(t)}]+[\hat{\varepsilon}_{is}^{(t)}-\varepsilon_{is}^{(t)}]\varepsilon_{js}^{(t)}+\varepsilon_{is}^{(t)}[\hat{\varepsilon}_{js}^{(t)}-\varepsilon_{js}^{(t)}]\}\equiv\sum_{l=1}^{3}c_{l,ij}^{(t)}, \end{aligned}$$

we have

$$
\begin{aligned}
Q_{2,213b}^{(t)} &= \frac{1}{N^2}\sum_{i,j=1}^{N}\frac{1}{T}X_i^{(t)\prime}F^{(t)}[\frac{1}{T}F^{(t)\prime}F^{(t)}]^{-1}(\frac{1}{N}\Lambda_t'\Lambda_t)^{-1}H^{(t)}\left[\hat{\lambda}_{jt}-H^{(t)-1}\lambda_{jt}\right][\hat{\varepsilon}^{(t)}(i,j)-\varepsilon^{(t)}(i,j)] \\
&= \sum_{l=1}^{3}\frac{1}{N^2}\sum_{i,j=1}^{N}\frac{1}{T}X_i^{(t)\prime}F^{(t)}[\frac{1}{T}F^{(t)\prime}F^{(t)}]^{-1}(\frac{1}{N}\Lambda_t'\Lambda_t)^{-1}H^{(t)}\left[\hat{\lambda}_{jt}-H^{(t)-1}\lambda_{jt}\right]c_{l,ij}^{(t)} \equiv \sum_{l=1}^{3}Q_{2,213}^{(t)}(l).
\end{aligned}
$$

Using

$$
\begin{aligned}
&\hat{\varepsilon}_{is}^{(t)}-\varepsilon_{is}^{(t)} \\
=\ & -X_{is}^{(t)\prime}\hat{\delta}_s-(\check{F}_s^{(t)}-H^{(t)\prime}F_s^{(t)})'H^{(t)-1}\lambda_{is}^{(t)}-(\check{F}_s^{(t)}-H^{(t)\prime}F_s^{(t)})'(\hat{\lambda}_{is}-H^{(t)-1}\lambda_{is})-F_s^{(t)\prime}H^{(t)}(\hat{\lambda}_{is}-H^{(t)-1}\lambda_{is}) \\
\equiv\ & \sum_{l=1}^{4}e_{is}^{(t)}(l),
\end{aligned}
$$

we can readily show that

$$
\begin{aligned}
&\max_{i,j,t}\left|\frac{1}{T}\sum_{s=1}^{T}[\hat{\varepsilon}_{is}^{(t)}-\varepsilon_{is}^{(t)}]\varepsilon_{js}^{(t)}\right| \\
\leq\ & \max_{i,j,t}\left|\frac{1}{T}\sum_{s=1}^{T}\varepsilon_{js}^{(t)}X_{is}^{(t)\prime}\hat{\delta}_s\right|+\max_{i,j,t}\left|\frac{1}{T}\sum_{s=1}^{T}(\check{F}_s^{(t)}-H^{(t)\prime}F_s^{(t)})'H^{(t)-1}\lambda\varepsilon_{js}^{(t)}\right| \\
&+\max_{i,j,t}\left|\frac{1}{T}\sum_{s=1}^{T}(\check{F}_s^{(t)}-H^{(t)\prime}F_s^{(t)})'(\hat{\lambda}_{is}-H^{(t)-1}\lambda_{is})\varepsilon_{js}^{(t)}\right|+\max_{i,j,t}\left|\frac{1}{T}\sum_{s=1}^{T}F_s^{(t)\prime}H^{(t)}(\hat{\lambda}_{is}-H^{(t)-1}\lambda_{is})\varepsilon_{js}^{(t)}\right| \\
=\ & O_P\left((N^{-1}+(Th)^{-1})\ln T+h^4\right)
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
&\max_t\frac{1}{T}\sum_{s=1}^{T}\left\|\frac{1}{NT}\sum_{i=1}^{N}[\hat{\varepsilon}_{is}^{(t)}-\varepsilon_{is}^{(t)}]\omega' X_i^{(t)\prime}F^{(t)}\right\| \\
=\ & \max_t\frac{1}{T}\sum_{s=1}^{T}\left\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{r=1}^{T}k_{h,rt}^{*}[\hat{\varepsilon}_{is}^{(t)}-\varepsilon_{is}^{(t)}]\omega' X_{ir}F_r\right\|=O_P\left((N^{-1}+(Th)^{-1})\ln T+h^2\right).
\end{aligned}
$$

With the first result, it is easy to show that $\max_t\left\|Q_{2,213b}^{(t)}(l)\right\| = o_P((NTh)^{-1/2})$ for $l=1,2,3$. Then $\max_t\left\|Q_{2,213b}^{(t)}\right\| = o_P((NTh)^{-1/2})$ and $\max_t\left\|Q_{2,213}^{(t)}\right\| = o_P((NTh)^{-1/2})$.

For $Q_{2,214}^{(t)}$, we have

$$
\begin{aligned}
Q_{2,214}^{(t)} &= \frac{1}{N^2}\sum_{i,j=1}^{N}\frac{1}{T}X_i^{(t)\prime}F^{(t)}[\frac{1}{T}F^{(t)\prime}F^{(t)}]^{-1}(\frac{1}{N}\Lambda_t'\Lambda_t)^{-1}\lambda_{jt}\left[\hat{\varepsilon}^{(t)}(i,j)-\varepsilon^{(t)}(i,j)\right] \\
&= \sum_{l=1}^{3}\frac{1}{N^2}\sum_{i,j=1}^{N}\frac{1}{T}X_i^{(t)\prime}F^{(t)}[\frac{1}{T}F^{(t)\prime}F^{(t)}]^{-1}(\frac{1}{N}\Lambda_t'\Lambda_t)^{-1}\lambda_{jt}c_{l,ij}^{(t)} \equiv \sum_{l=1}^{3}Q_{2,214}^{(t)}(l).
\end{aligned}
$$

We can show that $\max_t\left\|Q_{2,214}^{(t)}\right\| = o_P((NTh)^{-1/2})$ by showing that $\max_t\left\|Q_{2,214}^{(t)}(l)\right\| = o_P((NTh)^{-1/2})$ for

$l = 1, 2, 3$. For example, note that

$$\begin{aligned}
& \max_t \left| \omega' Q_{2,214}^{(t)}(2) \right| \\
= & \max_t \left| \operatorname{tr}\left([\frac{1}{T} F^{(t)\prime} F^{(t)}]^{-1} (\frac{1}{N} \Lambda_t' \Lambda_t)^{-1} \frac{1}{T} \sum_{s=1}^{T} \frac{1}{N} \sum_{j=1}^{N} \lambda_{jt} \varepsilon_{js}^{(t)} \frac{1}{N} \sum_{i=1}^{N} [\hat{\varepsilon}_{is}^{(t)} - \varepsilon_{is}^{(t)}] \frac{1}{T} \omega' X_i^{(t)\prime} F^{(t)} \right) \right| \\
\lesssim & \max_t \left\| \frac{1}{N^2 T^2} \sum_{s=1}^{T} \sum_{r=1}^{T} \sum_{j=1}^{N} k_{h,rt}^{*} \lambda_{jt} \varepsilon_{js}^{(t)} \sum_{i=1}^{N} [\hat{\varepsilon}_{is}^{(t)} - \varepsilon_{is}^{(t)}] \omega' X_{ir} F_r \right\| \\
\leq & \sum_{l=1}^{4} \max_t \left\| \frac{1}{N^2 T^2} \sum_{s=1}^{T} \sum_{r=1}^{T} \sum_{j=1}^{N} k_{h,rt}^{*} \lambda_{jt} \varepsilon_{js}^{(t)} \sum_{i=1}^{N} e_{is}^{(t)}(l)\, \omega' X_{ir} F_r \right\| \equiv \sum_{l=1}^{4} E_l,
\end{aligned}$$

and we can show that $E_l = o_P((NTh)^{-1/2}$ for $l = 1, 2, 3, 4$. In sum, we have $\max_t \left\| Q_{2,21}^{(t)} \right\| = o_P((NTh)^{-1/2})$.

Now, we consider the $Q_{2,22}^{(t)}$ term. The only difference between $Q_{2,21}^{(t)}$ and $Q_{2,22}^{(t)}$ is that $X_i^{(t)}$ is replaced by $\hat{V}_i^{(t)}$. The proof will be virtually the same if $\hat{V}_i^{(t)}$ is substituted by $V_i^{(t)}$. It is easy to argue that this substitution does not cause any difficulty in the analysis of $Q_{2,22}^{(t)}$. To see this, note that $\hat{V}_i^{(t)} - V_i^{(t)} = N^{-1} \sum_{k=1}^{N} (\hat{a}_{ik}^{(t)} - a_{ik}^{(t)}) X_k^{(t)}$, where

$$\begin{aligned}
\hat{a}_{ik}^{(t)} - a_{ik}^{(t)} = {} & [\hat{\lambda}_{it} - H^{(t)-1} \lambda_{it}]' (N^{-1} \hat{\Lambda}_t' \hat{\Lambda}_t)^{-1} \hat{\lambda}_{kt} + \lambda_{it}' H^{(t)\prime -1} \left[(N^{-1} \hat{\Lambda}_t' \hat{\Lambda}_t)^{-1} - H^{(t)\prime} (N^{-1} \Lambda_t' \Lambda_t)^{-1} H^{(t)} \right] \hat{\lambda}_{kt} \\
& + \lambda_{it}' (N^{-1} \Lambda_t' \Lambda_t)^{-1} H^{(t)} (\hat{\lambda}_{kt} - H^{(t)-1} \lambda_{kt}). \qquad \text{(A.11)}
\end{aligned}$$

Using Theorem 3.5(ii) below, we can readily show that $\max_{i,k,t} \left| \hat{a}_{ik}^{(t)} - a_{ik}^{(t)} \right| = O_P((Th/\ln T)^{-1/2} + h^2)$, which implies that $\max_{i,s,t} \left| \hat{V}_{is}^{(t)} - V_{is}^{(t)} \right| = O_P((Th/\ln T)^{-1/2} + h^2)$. Using the above decomposition, we can also argue that

$$\max_t \left\| \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{T} (\hat{V}_i^{(t)} - V_i^{(t)})' F^{(t)} (\frac{1}{T} F^{(t)\prime} F^{(t)})^{-1} (\frac{1}{N} \Lambda_t' \Lambda_t)^{-1} \lambda_{jt} \frac{1}{T} \sum_{s=1}^{T} \varepsilon_{is}^{(t)} \varepsilon_{js}^{(t)} \right\| = o_P((NTh)^{-1/2})$$

and similarly for other differences arising by replacing $V_i^{(t)}$ by $\hat{V}_i^{(t)}$. Then $\max_t \left\| Q_{2,22}^{(t)} \right\| = o_P((NTh)^{-1/2})$ and $\max_t \left\| Q_{2,2}^{(t)} \right\| = o_P((NTh)^{-1/2})$.

Now, we consider the $Q_{2,3}^{(t)}$ term. Note that $\hat{B}_{2\beta}^{(t)}$ is analogous to $\hat{\mathcal{B}}_{3NT}$ defined in Su and Ju (2018, SJ hereafter) and the only difference is that both the residual $\hat{\varepsilon}_{ir}^{(t)}$ and the regressor $X_{i,s}^{(t)}$ are now kernel-weighted in this paper and they are non-weighted in SJ. Following the proof of consistency of $\hat{\mathcal{B}}_{3NT}$ in that of Corollary 3.9 of SJ, we can a,so show that $\max_t \left\| Q_{2,3}^{(t)} \right\| = \max_t \frac{1}{Th} \left\| B_{2\beta}^{(t)} - \hat{B}_{2\beta}^{(t)} \right\| = o_P\left((NTh)^{-1/2} \right)$.

In sum, we have shown that $\max_t \left\| Q_2^{(t)} \right\| = o_P((NTh)^{-1/2})$.

Note that $\hat{B}_{4\beta}^{(t)}$ is analogous to $\hat{\mathcal{B}}_{1NT}$ defined in SJ. There are two main differences: 1) both the residual $\hat{\varepsilon}_{ir}^{(t)}$ and the regressor $X_{i,s}^{(t)}$ are now kernel weighted while they are non-weighted in SJ, and 2) the definitions in SJ use the fact that the error terms are serially uncorrelated conditional on the common sigma-field $\mathcal{C}$ while we do not assume so. Despite these differences, we can follow SJ's proofs of Corollary 3.9 closely and show that

$$\max_t \left\| Q_4^{(t)} \right\| = \max_t \frac{1}{Th} \left\| B_{4\beta}^{(t)} - \hat{B}_{4\beta}^{(t)} \right\| = o_P((NTh)^{-1/2}).$$

By Lemma A.7, $Q_3^{(t)} = \frac{1}{NT}\sum_{i=1}^N \tilde{Z}_i^{(t)\prime}\Delta_i^{(t)} = B_{1\beta}^{(t)} + o_P\left((NTh)^{-1/2}\right)$. It is easy to show that $\max_t \left\|B_{1\beta}^{(t)}\right\| = O_P(h^2) = O_P((NTh/\ln T)^{-1/2})$ when $h = O((NT)^{-1/5})$.

It follows that $\max_t \left\|\hat{\beta}_t^{bc} - \hat{\beta}_t\right\| = O_P((NTh/\ln T)^{-1/2})$ and $\sqrt{NTh}(\hat{\beta}_t^{bc} - \hat{\beta}_t - [D^{(t)}\left(F^{(t)}\right)]^{-1}B_{1\beta}^{(t)})$ is asymptotically normal with mean zero and variance-covariance matrix given in Theorem 3.2. ■

Proof of Theorem 3.5. Given the fast convergence rate of $\hat{\beta}_t^{bc}$ in Theorem 3.4, the proof is almost the same as that of Theorem 2.5 in Su and Wang (2020) and is thus omitted. ■

Proof of Theorem 3.6. The proof is almost the same as that of Theorem 3.6 in Su and Wang (2017), and is thus omitted. ■

B Proofs of Theorems in Section 4

To prove Theorems 4.1 and 4.3, we need the following lemma from Su and Chen (2013, Lemma A.5). It is also a conditional version of Lemma 2.1 in Sun and Chiang (1997).

Lemma B.1 *Let $\{\xi_t, t \geq 1\}$ be an l-dimensional strong mixing process conditional on $\mathcal{C}$ with mixing coefficient $\alpha_{\mathcal{C}}(\cdot)$ and distribution function $F_t(\cdot|\mathcal{C})$. Let the integers $(t_1, ..., t_m)$ be such that $1 \leq t_1 < t_2 < \cdots < t_m \leq T$. Suppose that $\max\{\int |\vartheta(v_1, ..., v_m)|^{1+\tilde{\eta}}\, dF_{t_1,...,t_m}(v_1, ..., v_m|\mathcal{C}), \int |\theta(v_1, ..., v_m)|^{1+\tilde{\eta}}\, dF_{t_1,...,t_j}(v_1, ..., v_j|\mathcal{C})$ $dF_{t_{j+1},...,t_m}(v_{j+1}, ..., v_m|\mathcal{C})\} \leq C_{\mathcal{C}}(t_1, ..., t_m)$ for some $\tilde{\eta} > 0$, where, e.g., $F_{t_1,...,t_m}(v_1, ..., v_m|\mathcal{C})$ denotes the distribution function of $\left(\xi_{t_1}, ..., \xi_{t_m}\right)$ given $\mathcal{C}$. Then*

$$\begin{aligned} & \left|\int \theta(v_1, ..., v_m)\, dF_{t_1,...,t_m}(v_1, ..., v_m|\mathcal{C}) - \int \theta(v_1, ..., v_m)\, dF^{(1)}_{t_1,...,t_j}(v_1, ..., v_j|\mathcal{C})\, dF_{t_{j+1},...,t_m}(v_{j+1}, ..., v_m)\right| \\ \leq\ & 4C_{\mathcal{C}}(t_1, ..., t_m)^{1/(1+\tilde{\eta})}\, \alpha_{\mathcal{C}}(t_{j+1} - t_j)^{\tilde{\eta}/(1+\tilde{\eta})}. \end{aligned}$$

Proof of Theorem 4.1 (i) For the convenience of proving Theorem 4.3 below, we prove that under $\mathbb{H}_A^{(1)}(a_{1NT})$ with $a_{1NT} = N^{-1/4}T^{-1/2}h^{-1/4}$, we have

$$J_{NT}^{(1)} = TN^{1/2}h^{1/2}\hat{M}^{(1)} - \mathbb{B}_{NT}^{(1)} - \Pi^{(1)} \xrightarrow{d} N(0, \mathbb{V}_0^{(1)}).$$

Noting that $\hat{\lambda}_{it}^{\prime}\hat{F}_t = (\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it})'H^{(t)\prime}F_t + \lambda_{it}'(H^{(t)-1})'(\hat{F}_t - H^{(t)\prime}F_t) + (\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it})'(\hat{F}_t - H^{(t)\prime}F_t) + \lambda_{it}'F_t$ and $\tilde{\lambda}_{i0}^{\prime}\tilde{F}_t = (\tilde{\lambda}_{i0} - H^{-1}\lambda_{i0})'H'F_t + \lambda_{i0}'\left(H^{-1}\right)'(\tilde{F}_t - H'F_t) + (\tilde{\lambda}_{i0} - H^{-1}\lambda_{i0})'(\tilde{F}_t - H'F_t) + \lambda_{i0}'F_t$, we have $\hat{\lambda}_{it}^{\prime}\hat{F}_t - \tilde{\lambda}_{i0}^{\prime}\tilde{F}_t = d_{1it} + d_{2it} + d_{3it}$, where

$$\begin{aligned} d_{1it} &= F_t'H^{(t)}(\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it}) - F_t'H(\tilde{\lambda}_{i0} - H^{-1}\lambda_{i0}) + \lambda_{it}'(H^{(t)-1})'(\hat{F}_t - H^{(t)\prime}F_t) - \lambda_{i0}'(H^{-1})'(\tilde{F}_t - H'F_t), \\ d_{2it} &= (\lambda_{it} - \lambda_{i0})'F_t, \text{ and} \\ d_{3it} &= (\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it})'(\hat{F}_t - H^{(t)\prime}F_t) - (\tilde{\lambda}_{i0} - H^{-1}\lambda_{i0})'(\tilde{F}_t - H'F_t). \end{aligned}$$

Let $d_{4it} = X_{it}'(\hat{\beta}_t^{bc} - \tilde{\beta})$. Then

$$\begin{aligned} & TN^{1/2}h^{1/2}\hat{M}^{(1)} = N^{-1/2}h^{1/2}\sum_{i=1}^N\sum_{t=1}^T [d_{1it} + d_{2it} + d_{3it} + d_{4it}]^2 \\ &= N^{-1/2}h^{1/2}\sum_{i=1}^N\sum_{t=1}^T \left[d_{1it}^2 + d_{2it}^2 + d_{3it}^2 + 2d_{1it}d_{2it} + 2d_{1it}d_{3it} + 2d_{2it}d_{3it} + d_{4it}^2 + 2d_{1it}d_{4it} + 2d_{2it}d_{4it} + 2d_{3it}d_{4it}\right] \\ &\equiv M_1 + M_2 + M_3 + 2M_4 + 2M_5 + 2M_6 + M_7 + 2M_8 + 2M_9 + 2M_{10}. \end{aligned}$$

The terms M_1 to M_6 have been analyzed by Su and Wang (2017, 2020). Following their analyses, we can also show that $M_2 = \Pi_2^{(1)} + o_P(1)$, and $M_j = o_P(1)$ for $j = 3, 4, 5, 6$. It remains to show that (i1) $M_1 - \mathbb{B}_{NT}^{(1)} - \Pi_1^{(1)} \xrightarrow{d} N(0, \mathbb{V}_0^{(1)})$, (i2) $M_7 = \Pi_3^{(1)} + o_P(1)$, and (i3) $M_j = o_P(1)$ for $j = 8, 9, 10$.

First, we show (i1). The asymptotic bias and variance of M_1 are different from that of the corresponding object in Su and Wang (2017) because we do not have the m.d.s. assumption on the error term. As in Su and Wang (2017), let $L_{st} = k_{h,st}^* H^{(t)} H^{(t)\prime} - HH'$ and $\bar{L}_{st} = (k_{h,st}^* - 1) H_0 H_0'$, where H_0 is the probability limit of both $H^{(t)}$ and H under the local alternative. By Lemma B.1(ii) and B.2(vi) in Su and Wang (2017),

$$\max_t \left\| H^{(t)} - H_0 \right\| = O_P(C_{NT}^{-1} (\ln T)^{1/2}) \text{ and } \|H - H_0\| = O_P(N^{-1/2} + T^{-1/2}). \tag{B.1}$$

Following their analysis, we can show that the leading term in d_{1it} is given by $d_{1it,1} + d_{1it,2}$ such that $M_1 = M_{1,1} + M_{1,2} + o_P(1)$, where $d_{1it,1} \equiv F_t' \frac{1}{T} \sum_{s=1}^T L_{st} F_s \varepsilon_{is}$, $d_{1it,2} \equiv a_{1NT} [F_t' H H' \frac{1}{T} \sum_{s=1}^T F_s g_{is}^\dagger + \lambda_{i0}' \left(H^{-1}\right)' V_{NT}^{-1}$ $\times (\frac{1}{T} \tilde{F}' F) \Lambda_0' g_t^\dagger / N]$, and $M_{1,l} = N^{-1/2} h^{1/2} \sum_{i=1}^N \sum_{t=1}^T d_{1it,l}^2$ for $l = 1, 2$. Following Su and Wang (2017), we can show that $M_{1,2} = \Pi_1^{(1)} + o_P(1)$. Let $\bar{L}_{st} = (k_{h,st}^* - 1) H_0 H_0'$. For $M_{1,1}$, we can decompose it as follows:

$$\begin{aligned}
M_{1,1} &= \frac{h^{1/2}}{N^{1/2} T^2} \sum_{i=1}^N \sum_{t,s,r=1}^T F_t' L_{st} F_s F_r' L_{rt}' F_t \varepsilon_{is} \varepsilon_{ir} \\
&= \frac{h^{1/2}}{N^{1/2} T^2} \sum_{i=1}^N \sum_{t,s,r=1}^T \left\{ F_t' L_{st} F_s F_r' L_{rt}' F_t E_{\mathcal{C}}(\varepsilon_{i,sr}) + F_t' \bar{L}_{st} F_s F_r' \bar{L}_{rt}' F \tilde{\varepsilon}_{i,sr} \right. \\
&\quad \left. + 2 F_t' (L_{st} - \bar{L}_{st}) F_s F_r' \bar{L}_{rt} F_t \tilde{\varepsilon}_{i,sr} + F_t' (L_{st} - \bar{L}_{st}) F_s F_r' (L_{rt} - \bar{L}_{rt})' F_t \tilde{\varepsilon}_{i,sr} \right\} \equiv \sum_{\ell=1}^4 M_{1,1}^{(\ell)},
\end{aligned}$$

where $\varepsilon_{i,sr} \equiv \varepsilon_{is} \varepsilon_{ir}$ and $\tilde{\varepsilon}_{i,sr} = \varepsilon_{is} \varepsilon_{ir} - E_{\mathcal{C}}(\varepsilon_{is} \varepsilon_{ir})$. Note that $M_{1,1}^{(1)} = \mathbb{B}_{NT}^{(1)}$. Following the analysis of $M_{1,1}^{(2)}$ below, we can readily show that $M_{1,1}^{(l)} = o_P(1)$ for $l = 3, 4$. For $M_{1,1}^{(2)}$, we can further decompose it as follows

$$\begin{aligned}
M_{1,1}^{(2)} &= \frac{h^{1/2}}{N^{1/2} T} \sum_{i=1}^N \sum_{s,r=1}^T \operatorname{tr}\{F_s F_r' \frac{1}{T} \sum_{t=1}^T \bar{L}_{rt}' F_t F_t' \bar{L}_{st}\} \tilde{\varepsilon}_{i,sr} \\
&= \frac{h^{1/2}}{N^{1/2} T} \sum_{i=1}^N \sum_{s,r=1}^T \frac{1}{T} \sum_{t=1}^T \operatorname{tr}\left\{ F_s F_r' H_0 H_0' \left(k_{h,st}^* k_{h,rt}^* - 2 k_{h,st} + 1 \right) F_t F_t' H_0 H_0' \right\} \tilde{\varepsilon}_{i,sr} \\
&\equiv M_{1,1}^{(2,1)} - 2 M_{1,1}^{(2,2)} + M_{1,1}^{(2,3)}, \text{ say.}
\end{aligned}$$

We will show that $M_{1,1}^{(2,1)}$ is $O_P(1)$ and contributes to the asymptotic variance and similar analyses show that $M_{1,1}^{(2,2)}$ and $M_{1,1}^{(2,3)}$ are both $O_P\left(h^{1/2}\right)$. Let $\xi_{h,sr} \equiv F_r' H_0 H_0' \frac{1}{T} \sum_{t=1}^T k_{h,st}^* k_{h,rt}^* F_t F_t' H_0 H_0' F_s$. Note that $\xi_{h,sr}$ is $\mathcal{C}$-measurable and

$$M_{1,1}^{(2,1)} = \sum_{i=1}^N \frac{h^{1/2}}{N^{1/2} T} \sum_{s,r=1}^T \xi_{h,sr} \tilde{\varepsilon}_{i,sr} \equiv \sum_{i=1}^N Z_{NT,i},$$

where $Z_{NT,i} = \frac{h^{1/2}}{N^{1/2} T} \sum_{s,r=1}^T \xi_{h,sr} \tilde{\varepsilon}_{i,sr}$. By the CLT for independent but non-identically distributed variables (conditional on $\mathcal{C}$), it suffices to prove $\mathbb{V}_{NT}^{-1/2} M_{1,1}^{(2,1)} \xrightarrow{d} N(0,1)$ by showing that

$$\mathcal{Z} \equiv \sum_{i=1}^N E_{\mathcal{C}}\left(Z_{NT,i}^4 \right) = o_P(1) \text{ and } \sum_{i=1}^N Z_{NT,i}^2 - \mathbb{V}_{NT}^{(1)} = o_P(1). \tag{B.2}$$

First, we verify the first part of (B.2). Note that

$$\mathcal{Z} \leq \frac{16h^2}{N^2T^4}\sum_{i=1}^{N} E_{\mathcal{C}}\left[\sum_{s=1}^{T}\xi_{h,ss}\tilde{\varepsilon}_{i,ss}\right]^4 + \frac{256h^2}{N^2T^4}\sum_{i=1}^{N} E_{\mathcal{C}}\left[\sum_{1\leq s<r\leq T}\xi_{h,sr}\tilde{\varepsilon}_{i,sr}\right]^4 \equiv 16\mathcal{Z}_1 + 256\mathcal{Z}_2.$$

For $\mathcal{Z}_1$, we have

$$\begin{aligned}
\mathcal{Z}_1 &= \frac{h^2}{N^2T^4}\sum_{i=1}^{N}\sum_{s_l\in[T],l=1:4}\xi_{h,s_1s_1}\xi_{h,s_2s_2}\xi_{h,s_3s_3}\xi_{h,s_4s_4}E_{\mathcal{C}}(\tilde{\varepsilon}_{i,s_1s_1}\tilde{\varepsilon}_{i,s_2s_2}\tilde{\varepsilon}_{i,s_3s_3}\tilde{\varepsilon}_{i,s_4s_4})\\
&\lesssim \frac{h^2}{N^2T^4}\sum_{i=1}^{N}\Bigg[\sum_{s=1}^{T}\xi_{h,ss}^4E_{\mathcal{C}}(\tilde{\varepsilon}_{i,ss}^4) + \sum_{1\leq s_1\neq s_2\leq T}\xi_{h,s_1s_1}^2\xi_{h,s_2s_2}^2E_{\mathcal{C}}(\tilde{\varepsilon}_{i,s_1s_1}^2\tilde{\varepsilon}_{i,s_2s_2}^2)\\
&+\sum_{1\leq s_1\neq s_2\leq T}\xi_{h,s_1s_1}^3\xi_{h,s_2s_2}E_{\mathcal{C}}(\tilde{\varepsilon}_{i,s_1s_1}^3\tilde{\varepsilon}_{i,s_2s_2}) + \sum_{1\leq s_1\neq s_2\neq s_3\leq T}\xi_{h,s_1s_1}^2\xi_{h,s_2s_2}\xi_{h,s_3s_3}E_{\mathcal{C}}(\tilde{\varepsilon}_{i,s_1s_1}^2\tilde{\varepsilon}_{i,s_2s_2}\tilde{\varepsilon}_{i,s_3s_3})\\
&+\sum_{1\leq s_1\neq s_2\neq s_3\neq s_4\leq T}\xi_{h,s_1s_1}\xi_{h,s_2s_2}\xi_{h,s_3s_3}\xi_{h,s_4s_4}E_{\mathcal{C}}(\tilde{\varepsilon}_{i,s_1s_1}\tilde{\varepsilon}_{i,s_2s_2}\tilde{\varepsilon}_{i,s_3s_3}\tilde{\varepsilon}_{i,s_4s_4})\Bigg]\\
&\equiv \mathcal{Z}_{1,1}+\mathcal{Z}_{1,2}+\mathcal{Z}_{1,3}+\mathcal{Z}_{1,4}+\mathcal{Z}_{1,5}.
\end{aligned}$$

By straightforward moment calculations, we can show that $\mathcal{Z}_{1,1} = O_P(N^{-1}T^{-3}h^{-1})$, $\mathcal{Z}_{1,2} = O_P(N^{-1}T^{-2})$, $\mathcal{Z}_{1,3} = O_P(N^{-1}T^{-2})$, $\mathcal{Z}_{1,4} = O_P(N^{-1}T^{-1}h)$ and $\mathcal{Z}_{1,5} = O_P(N^{-1}h^2)$. It follows that $\mathcal{Z}_1 = o_P(1)$. Next,

$$\begin{aligned}
\mathcal{Z}_2 &= \frac{h^2}{N^2T^4}\sum_{i=1}^{N}E_{\mathcal{C}}\left[\sum_{1\leq s<r\leq T}\xi_{h,sr}\tilde{\varepsilon}_{i,sr}\right]^4\\
&\leq \frac{16h^2}{N^2T^4}\sum_{i=1}^{N}E_{\mathcal{C}}\left[\sum_{1\leq s<r\leq T}\xi_{h,sr}\varepsilon_{i,sr}\right]^4 + \frac{16h^2}{N^2T^4}\sum_{i=1}^{N}\left[\sum_{1\leq s<r\leq T}\xi_{h,sr}E_{\mathcal{C}}(\varepsilon_{i,sr})\right]^4 \equiv 16(\mathcal{Z}_{2,1}+\mathcal{Z}_{2,2}).
\end{aligned}$$

We first consider $\mathcal{Z}_{2,1} = \frac{h^2}{N^2T^4}\sum_{i=1}^{N}\sum_{s_l,r_l\in[T],\ s_l<r_l,l=1:4}\xi_{h,s_1r_1}\xi_{h,s_2r_2}\xi_{h,s_3r_3}\xi_{h,s_4r_4}E_{\mathcal{C}}(\varepsilon_{i,s_1r_1}\varepsilon_{i,s_2r_2}\varepsilon_{i,s_3r_3}\varepsilon_{i,s_4r_4})$. For the summation over s_l and r_l, $l\in[4]$, in the last expression, we consider seven cases depending on the cardinality (#) of the set $\mathcal{S}\equiv\{s_1,r_1,...,s_4,r_4\}$: $\#\mathcal{S} = 8,7,6,5,4,3,2$, and write $\mathcal{Z}_2 = \sum_{l=1}^{7}\mathcal{Z}_{2,l}$, where

$$\mathcal{Z}_{2,1l} = \frac{h^2}{N^2T^4}\sum_{i=1}^{N}\sum_{s_r,r_r\in[T],\ s_r<r_r,r=1:4,\#\mathcal{S}=9-l}\xi_{h,s_1r_1}\xi_{h,s_2r_2}\xi_{h,s_3r_3}\xi_{h,s_4r_4}E_{\mathcal{C}}(\varepsilon_{i,s_1r_1}\varepsilon_{i,s_2r_2}\varepsilon_{i,s_3r_3}\varepsilon_{i,s_4r_4} \text{ for } l\in[7].$$

We will show $\mathcal{Z}_{2,11} = o_P(1)$ as the other cases are similar or simpler. Now, we can rewrite $\mathcal{Z}_{2,11}$ as follows

$$\mathcal{Z}_{2,11} = \frac{h^2}{N^2T^4}\sum_{i=1}^{N}\sum_{s_1,...,s_8\in[T],\ s_1,...,s_8\text{ are distinct}}\xi_{h,s_1s_2}\xi_{h,s_3s_4}\xi_{h,s_5s_6}\xi_{h,s_7s_8}E_{\mathcal{C}}(\varepsilon_{is_1}\varepsilon_{is_2}...\varepsilon_{is_8}).$$

Let $1\leq r_1<r_2<\cdots<r_8\leq T$ be the permutation of $s_1,s_2,...,s_8$ in descending order and let d_c be the c-th largest difference among $r_{j+1}-r_j$, $j\in[7]$. For notational simplicity, let $L_i(r_1,...,r_8) = \xi_{h,s_1s_2}\xi_{h,s_3s_4}\xi_{h,s_5s_6}\xi_{h,s_7s_8}$ $\times E_{\mathcal{C}}(\varepsilon_{is_1}\varepsilon_{is_2}...\varepsilon_{is_8})$. Note that $\xi_{h,sr}\lesssim\|F_r\|\,\|F_s\|\,h^{-1}k^{**}(\frac{s-r}{h})$, where $k^{**}(\cdot)$ is the 2-fold convolution kernel of

$k^*(\cdot)$. Then by Lemma B.1 (with $\tilde{\eta} = \eta/2$)

$$\begin{aligned}
&\frac{h^2}{N^2T^4}\sum_{i=1}^{N}\sum_{1\le r_1<\cdots<r_8\le T,\ r_2-r_1=d_1}|L_i(r_1,...,r_8)| \\
\le\ & \frac{4M_{NT}h^2}{N^2T^4}\sum_{i=1}^{N}\sum_{r_1=1}^{T-7}\sum_{r_2=r_1+\max_{j\ge3}\{r_j-r_{j-1}\}}^{T-6}\sum_{r_3=k_2+1}^{T-5}\cdots\sum_{r_8=r_7+1}^{T}\left|\xi_{h,s_1s_2}\xi_{h,s_3s_4}\xi_{h,s_5s_6}\xi_{h,s_7s_8}\right|[\alpha_{\mathcal{C}}(r_2-r_1)]^{\frac{\eta}{2+\eta}} \\
\lesssim\ & \max_r\|F_r\|^7 h^{-4}\frac{4M_{NT}h^2}{NT^4}\sum_{r_1=1}^{T-7}\sum_{r_2=r_1+\max_{j\ge3}\{r_j-r_{j-1}\}}^{T-6}\|F_{r_1}\|(r_2-r_1)^6[\alpha_{\mathcal{C}}(r_2-r_1)]^{\frac{\eta}{2+\eta}} \\
\lesssim\ & \frac{T^{7/(8+4\eta)}h^{-2}}{NT^3}\sum_{r=1}^{\infty}r^6[\alpha_{\mathcal{C}}(r)]^{\frac{\eta}{2+\eta}} = O_P(N^{-1}T^{-3+7/(8+4\eta)}h^{-2}) = o_P(1),
\end{aligned}$$

where $M_{NT} = [\max_{i,t}E_{\mathcal{C}}(|\varepsilon_{it}|^{8+4\eta})]^{\eta/(2+\eta)} = O_P(1)$. Similarly, $\frac{h^2}{N^2T^4}\sum_{i=1}^{N}\sum_{1\le r_1<\cdots<r_8\le T,\ r_8-r_7=d_1}|L_i(r_1,...,r_8)| = o_P(1)$. If for some l_α with $2\le l_\alpha\le 6$ and $1\le\alpha\le4$, $r_{l_\alpha+1}-r_{l_\alpha}=d_\alpha$, then we can show that

$$\begin{aligned}
&\frac{h^2}{N^2T^4}\sum_{i=1}^{N}\sum_{1\le r_1<\cdots<r_8\le T,\ r_{l_\alpha+1}-r_{l_\alpha}=d_\alpha,\ 2\le l_\alpha\le6, 1\le\alpha\le4}|L_i(r_1,...,r_8)| \\
\lesssim\ & \frac{\max_r\|F_r\|^6h^{-1}}{NT^2}\sum_{r=1}^{\infty}r^5\alpha^{\frac{\eta}{2+\eta}}(r) + \frac{\max_r\|F_r\|^5}{NT}\sum_{r=1}^{\infty}r^4\alpha^{\frac{\eta}{2+\eta}}(r) + \frac{\max_r\|F_r\|^4h}{N}\sum_{r=1}^{\infty}r^3\alpha^{\frac{\eta}{2+\eta}}(r) \\
=\ & O_P(N^{-1}T^{-2+6/(8+4\eta)}h^{-1}) + O_P(N^{-1}T^{-1+5/(8+4\eta)}) + O_P(N^{-1}T^{1/(2+\eta)}h) = o_P(1).
\end{aligned}$$

In sum, $\mathcal{Z}_{2,11} = o_P(1)$. Analogously, $\mathcal{Z}_{2,1l} = o_P(1)$ for $l = 2,...,7$. It follows that $\mathcal{Z}_{2,1} = o_P(1)$. For $\mathcal{Z}_{2,2}$, in view of the fact that $\xi_{h,sr} \equiv F_r'H_0H_0'\frac{1}{T}\sum_{t=1}^{T}k_{h,st}^*k_{h,rt}^*F_tF_t'H_0H_0'F_s$ and $\bar{k}_{h,sr}^* = \frac{1}{T}\sum_{t=1}^{T}k_{h,st}^*k_{h,rt}^*$

$$\begin{aligned}
&\frac{1}{T}\left|\sum_{1\le s<r\le T}\xi_{h,sr}E_{\mathcal{C}}(\varepsilon_{i,sr})\right| \\
\asymp\ & \frac{1}{T}\left|\sum_{1\le s<r\le T}F_r'H_0H_0'\Sigma_FH_0H_0'F_s\frac{1}{T}\sum_{t=1}^{T}k_{h,st}^*k_{h,rt}^*E_{\mathcal{C}}(\varepsilon_{i,sr})\right| \\
=\ & \frac{1}{T}\left|\sum_{1\le s<r\le T}\bar{k}_{h,sr}^*F_r'H_0H_0'\Sigma_FH_0H_0'F_sE_{\mathcal{C}}(\varepsilon_{i,sr})\right| \lesssim \frac{1}{T}\sum_{s=1}^{T}\sum_{r=1}^{T}\bar{k}_{h,sr}^*\left(\|F_r\|^2+\|F_s\|^2\right)|E_{\mathcal{C}}(\varepsilon_{i,sr})| \\
\lesssim\ & \frac{1}{T}\sum_{r=1}^{T}\max_s\bar{k}_{h,sr}^*\|F_r\|^2\sum_{s=1}^{T}|E_{\mathcal{C}}(\varepsilon_{i,sr})| \lesssim \max_{i,r}\sum_{s=1}^{T}|E_{\mathcal{C}}(\varepsilon_{is}\varepsilon_{ir})|\frac{1}{T}\sum_{r=1}^{T}\max_s\bar{k}_{h,sr}^*\|F_r\|^2 = O_P(1),
\end{aligned}$$

we have $\mathcal{Z}_{2,2} = \frac{h^2}{N^2}\sum_{i=1}^{N}\left[\frac{1}{T}\sum_{1\le s<r\le T}\xi_{h,sr}E_{\mathcal{C}}(\varepsilon_{i,sr})\right]^4 = O_P(N^{-1}h^2) = o_P(1)$. Consequently, $\mathcal{Z}_2 = o_P(1)$ and $\mathcal{Z} = o_P(1)$.

Now, we verify the second part of (B.2). It suffices to show that (I) $\sum_{i=1}^{N}E_{\mathcal{C}}\left(Z_{NT,i}^2\right) = \mathbb{V}_{NT} + o_P(1)$, and (II) $\mathrm{Var}_{\mathcal{C}}\left(\sum_{i=1}^{N}Z_{NT,i}^2\right) = o_P(1)$ by conditional Chebyshev inequality. (I) follows because

$$\sum_{i=1}^{N}E_{\mathcal{C}}\left(Z_{NT,i}^2\right) = \frac{h}{NT^2}\sum_{i=1}^{N}E_{\mathcal{C}}\left(\sum_{s,r=1}^{T}\xi_{h,sr}\tilde{\varepsilon}_{i,sr}\right)^2$$

$$= \frac{h}{N}\sum_{i=1}^{N} E_{\mathcal{C}}\left(\frac{1}{T}\sum_{s,r=1}^{T}\xi_{h,sr}\varepsilon_{is}\varepsilon_{ir}\right)^2 - \frac{h}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{s,r=1}^{T}\xi_{h,sr}E_{\mathcal{C}}(\varepsilon_{is}\varepsilon_{ir})\right)^2 = \mathbb{V}_{NT}^{(1)} - o_P(1).$$

(II) follows because

$$\begin{aligned}\mathrm{Var}_{\mathcal{C}}\left(\sum_{i=1}^{N} Z_{NT,i}^2\right) &= \sum_{i=1}^{N}\mathrm{Var}_{\mathcal{C}}\left(Z_{NT,i}^2\right) = \frac{h^2}{N^2T^4}\sum_{i=1}^{N} E\left[\mathrm{Var}_{\mathcal{C}}\left(\sum_{s,r=1}^{T}\xi_{h,sr}\tilde{\varepsilon}_{i,sr}\right)^2\right] \le \frac{h^2}{N^2T^4}\sum_{i=1}^{N} E_{\mathcal{C}}\left|\sum_{s,r=1}^{T}\xi_{h,sr}\tilde{\varepsilon}_{i,sr}\right|^4 \\ &= o_P(1).\end{aligned}$$

In sum, we have shown that $M_{1,1}^{(2)} - \mathbb{B}_{NT}^{(1)} \xrightarrow{d} N(0, \mathbb{V}_0)$ and $M_1 - \mathbb{B}_{NT}^{(1)} - \Pi_1^{(1)} \xrightarrow{d} N(0, \mathbb{V}_0^{(1)})$.

Now, we show (i2) and (i3). Note that

$$d_{4it} = X_{it}'(\hat{\beta}_t^{bc} - \tilde{\beta}) = X_{it}'(\hat{\beta}_t^{bc} - \beta_t) - X_{it}'(\tilde{\beta} - \beta_0) + X_{it}'(\beta_t - \beta_0).$$

Apparently, $X_{it}'(\beta_t - \beta_0) = 0$ under $\mathbb{H}_0^{(1)}$, and $X_{it}'(\beta_t - \beta_0) = a_{1NT}X_{it}'g_{0t}$ under $\mathbb{H}_A^{(1)}(a_{1NT})$. Then, we can further decompose M_7 as follows

$$\begin{aligned}M_7 &= N^{-1/2}h^{1/2}\sum_{i=1}^{N}\sum_{t=1}^{T} d_{4it}^2 \\ &= N^{-1/2}h^{1/2}\sum_{i=1}^{N}\sum_{t=1}^{T}\Big\{[X_{it}'(\hat{\beta}_t^{bc} - \beta_t)]^2 + [X_{it}'(\tilde{\beta} - \beta_0)]^2 + [X_{it}'(\beta_t - \beta_0)]^2 \\ &\quad -2X_{it}'(\hat{\beta}_t^{bc} - \beta_t)(\tilde{\beta} - \beta_0)'X_{it} + 2X_{it}'(\hat{\beta}_t^{bc} - \beta_t)(\beta_t - \beta_0)'X_{it} - 2X_{it}'(\tilde{\beta} - \beta_0)(\beta_t - \beta_0)'X_{it}\Big\} \\ &\equiv M_{7,1} + M_{7,2} + M_{7,3} - 2M_{7,4} + 2M_{7,5} - 2M_{7,6}.\end{aligned}$$

By Theorem 3.4(ii), we have $\max_t \left\|\hat{\beta}_t^{bc} - \beta_t\right\| = O_P((NTh/\ln T)^{-1/2})$. Following the proof of Lemma A.1 in Su and Chen (2013), we can show that under $\mathbb{H}_A^{(1)}(a_{1NT})$,

$$\tilde{\beta} - \beta_0 = a_{1NT}D(F)^{-1}\pi_{NT} + o_P(a_{1NT}), \tag{B.3}$$

where $D(F)$ is defined in Section 4.4. Let $\pi_{NT} = (\pi_{NT,1}, ..., \pi_{NT,P})'$, $\pi_{NT,p} = \frac{1}{NT}\mathrm{tr}(M_F\mathbf{X}_pM_{\Lambda_0}\Delta_0')$ for $p = 1, ..., P$, and Δ_0 is a $T \times N$ matrix with (t,i)th element given by $X_{it}'g_{0t}$. With these results, we can readily show that $M_{7,1} = O_P((Nh)^{-1/2}\ln T) = o_P(1)$,

$$\begin{aligned}M_{7,2} &= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left[X_{it}'D(F)^{-1}\pi_{NT}\right]^2 + o_P(1),\ M_{7,3} = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}[X_{it}'g_{0t}]^2, \text{ and} \\ M_{7,6} &= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}X_{it}'D(F)^{-1}\pi_{NT}g_{0t}'X_{it} + o_P(1).\end{aligned}$$

Then $M_{7,4} \le \{M_{7,1}M_{7,2}\}^{1/2} = o_P(1)$ and $M_{7,5} \le \{M_{7,1}M_{7,3}\}^{1/2} = o_P(1)$ by CS inequality. Consequently, we have

$$\begin{aligned}M_7 &= M_{7,2} + M_{7,3} - 2M_{7,6} + o_P(1) \\ &= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left[X_{it}'(g_{0t} - D(F)^{-1}\pi_{NT})\right]^2 + o_P(1) = \Pi_3^{(1)} + o_P(1).\end{aligned}$$

Similarly, we can show that $M_l = o_P(1)$ for $l = 8$ and 9. For M_{10}, we have $M_{10} \leq \{M_3 M_4\}^{1/2} = o_P(1)$ by CS inequality.

In sum, we have shown that $J_{NT}^{(1)} \equiv TN^{1/2}h^{1/2}\hat{M}^{(1)} - \mathbb{B}_{NT}^{(1)} - \Pi^{(1)} \xrightarrow{d} N(0, \mathbb{V}_0^{(1)})$, where $\Pi^{(1)} = \sum_{\ell=1}^{3} \Pi_\ell^{(1)}$.

(ii) As in the proof of (i), we prove that under $\mathbb{H}_A^{(2)}(a_{2NT}) : \beta_t = \beta_0 + a_{2NT} g_{0t}$ with $a_{2NT} = (NT)^{-1/2}h^{-1/4}$, we have $J_{NT}^{(2)} = TNh^{1/2}\hat{M}^{(2)} - \mathbb{B}_{NT}^{(2)} \xrightarrow{d} N(0, \mathbb{V}_0^{(2)})$. First, we decompose $TNh^{1/2}\hat{M}^{(2)}$ as follows:

$$\begin{aligned} TNh^{1/2}\hat{M}^{(2)} &= Nh^{1/2}\sum_{t=1}^{T}\left\|(\hat{\beta}_t^{bc} - \beta_t) - (\bar{\hat{\beta}}^{bc} - \beta_t)\right\|^2 \\ &= Nh^{1/2}\sum_{t=1}^{T}\left\{\left\|\hat{\beta}_t^{bc} - \beta_t\right\|^2 + \left\|\bar{\hat{\beta}}^{bc} - \beta_t\right\|^2 - 2(\hat{\beta}_t^{bc} - \beta_t)'(\bar{\hat{\beta}}^{bc} - \beta_t)\right\} \equiv A_1 + A_2 - 2A_3. \end{aligned}$$

Following the proofs of Theorems 3.2 and 3.4, we can argue that $\hat{\beta}_t^{bc} - \beta_t = [D^{(t)}(F^{(t)})]^{-1}\frac{1}{NT}\sum_{i=1}^{N}\sum_{s=1}^{T}\mathfrak{X}_{is}^{(t)}\varepsilon_{is}^{(t)} + R_\beta(t)$ and $R_\beta(t) = o_P(a_{2NT})$ uniformly in t under $\mathbb{H}_A^{(2)}(a_{2NT})$ under the additional bandwidth condition $NTh^{9/2} = o(1)$. For A_1, we have

$$\begin{aligned} A_1 = Nh^{1/2}\sum_{t=1}^{T}&\left\{\left\|[D^{(t)}(F^{(t)})]^{-1}\frac{1}{NT}\sum_{i=1}^{N}\sum_{s=1}^{T}\mathfrak{X}_{is}^{(t)}\varepsilon_{is}^{(t)}\right\|^2 + \|R_\beta(t)\|^2 + 2\left[[D^{(t)}(F^{(t)})]^{-1}\frac{1}{NT}\sum_{i=1}^{N}\sum_{s=1}^{T}\mathfrak{X}_{is}^{(t)}\varepsilon_{is}^{(t)}\right]' R_\beta(t)\right\} \\ \equiv A_{1,1} &+ A_{1,2} + 2A_{1,3}. \end{aligned}$$

We will show that the first term is the leading term, while the second and third terms are higher order terms. Before that, we first analyze the $\mathfrak{X}_{is}^{(t)}$ term in detail. Note that the (s,r)th element of $M_{F^{(t)}}$ is

$$[M_{F^{(t)}}]_{s,r} = \mathbf{1}\{s = r\} - T^{-1}F_s^{(t)\prime}[T^{-1}F^{(t)\prime}F^{(t)}]^{-1}F_r^{(t)} = \mathbf{1}\{s = r\} - T^{-1}b_{sr}^{(t)},$$

and the (i,j)th element of M_{Λ_t} is given by $[M_{\Lambda_t}]_{i,j} = \mathbf{1}\{i = j\} - N^{-1}a_{ij}^{(t)}$, where $a_{ij}^{(t)} = \lambda_{it}'(\Lambda_t'\Lambda_t/N)^{-1}\lambda_{jt}$ and $b_{sr}^{(t)} = F_s^{(t)\prime}(T^{-1}\sum_{l=1}^{T}F_l^{(t)\prime}F_l^{(t)})^{-1}F_r^{(t)}$. Let $\bar{b}_{sr}^{(t)} = F_s'[T^{-1}\sum_{l=1}^{T}k_{h,lt}^{*}F_lF_l']^{-1}F_r$. The (s,i)th element of the $T \times N$ matrix $\mathfrak{X}^{(t)}$ is given by

$$\begin{aligned} \mathfrak{X}_{is}^{(t)} &= \sum_{j=1}^{N}\sum_{r=1}^{T}k_{h,rt}^{*1/2}[M_{F^{(t)}}]_{s,r}E_{\mathcal{C}}(X_{jr})\,[M_{\Lambda_t}]_{j,i} + [X_{is}^{(t)} - E_{\mathcal{C}}(X_{is}^{(t)})] \\ &= \sum_{j=1}^{N}\sum_{r=1}^{T}k_{h,rt}^{*1/2}\left[1\,(s = r) - T^{-1}b_{sr}^{(t)}\right]E_{\mathcal{C}}(X_{jr})\left[1\,(i = j) - N^{-1}a_{ij}^{(t)}\right] + [X_{is}^{(t)} - E_{\mathcal{C}}(X_{is}^{(t)})] \\ &= X_{is}^{(t)} - \frac{1}{N}\sum_{j=1}^{N}k_{h,st}^{*1/2}E_{\mathcal{C}}(X_{jr})a_{ij}^{(t)} - \frac{1}{T}\sum_{r=1}^{T}k_{h,rt}^{*1/2}b_{sr}^{(t)}E_{\mathcal{C}}(X_{jr}) + \frac{1}{NT}\sum_{j=1}^{N}\sum_{r=1}^{T}k_{h,rt}^{*1/2}b_{sr}^{(t)}E_{\mathcal{C}}(X_{jr})a_{ij}^{(t)} \\ &= k_{h,st}^{*1/2}\left[X_{is} - \frac{1}{N}\sum_{j=1}^{N}E_{\mathcal{C}}(X_{js})a_{ji}^{(t)} - \frac{1}{T}\sum_{r=1}^{T}k_{h,rt}^{*}\bar{b}_{sr}^{(t)}E_{\mathcal{C}}(X_{jr}) + \frac{1}{NT}\sum_{r=1}^{T}\sum_{j=1}^{N}k_{h,rt}^{*}\bar{b}_{sr}^{(t)}a_{ji}^{(t)}E_{\mathcal{C}}(X_{jr})\right] \\ &\equiv k_{h,st}^{*1/2}\mathcal{X}_{is}^{(t)}. \end{aligned}$$

For $A_{1,1}$, we can readily show that

$$A_{1,1} = Nh^{1/2}\sum_{t=1}^{T}\frac{1}{N^2T^2}\sum_{i,j=1}^{N}\sum_{s,r=1}^{T}\varepsilon_{is}^{(t)}\varepsilon_{jr}^{(t)}\mathfrak{X}_{is}^{(t)\prime}\mathbb{D}^{(t)}\mathfrak{X}_{jr}^{(t)}$$

$$= \frac{h^{1/2}}{NT^2} \sum_{t=1}^{T} \sum_{i=1}^{N} \sum_{s,r=1}^{T} k_{h,st}^* k_{h,rt}^* \varepsilon_{is} \varepsilon_{ir} \mathcal{X}_{is}^{(t)\prime} \mathbb{D}^{(t)} \mathfrak{X}_{ir}^{(t)} + \frac{2h^{1/2}}{NT^2} \sum_{1 \le j < i \le N} \sum_{t=1}^{T} \sum_{s=1}^{T} k_{h,st}^{*2} \varepsilon_{is} \varepsilon_{js} \mathcal{X}_{is}^{(t)\prime} \mathbb{D}^{(t)} \mathcal{X}_{js}^{(t)}$$

$$+ \frac{2h^{1/2}}{NT^2} \sum_{1 \le j < i \le N} \sum_{t=1}^{T} \sum_{s \ne r} k_{h,st}^* k_{h,rt}^* \varepsilon_{is} \varepsilon_{jr} \mathcal{X}_{is}^{(t)\prime} \mathbb{D}^{(t)} \mathcal{X}_{jr}^{(t)} \equiv \sum_{\ell=1}^{3} A_{1,1\ell},$$

where $\mathbb{D}^{(t)} = [D^{(t)}(F^{(t)})]^{-1}[D^{(t)}(F^{(t)})]^{-1}$ and $\sum_{s \ne r} = \sum_{1 \le s \ne r \le T}$. We will show that $A_{1,11}$ and $A_{1,12}$ contribute to the asymptotic bias and $A_{1,13}$ contributes the asymptotic variance. Apparently,

$$A_{1,11} = \frac{h^{1/2}}{NT^2} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s,r=1}^{T} k_{h,st}^* k_{h,rt}^* \varepsilon_{is} \varepsilon_{ir} \mathcal{X}_{is}^{(t)\prime} \mathbb{D}^{(t)} \mathcal{X}_{ir}^{(t)} = \mathbb{B}_{1,NT}^{(2)} \text{ and}$$

$$A_{1,12} = \frac{2h^{1/2}}{NT^2} \sum_{1 \le j < i \le N} \sum_{t=1}^{T} \sum_{s=1}^{T} k_{h,st}^{*2} \varepsilon_{is} \varepsilon_{js} \mathcal{X}_{is}^{(t)\prime} \mathbb{D}^{(t)} \mathcal{X}_{js}^{(t)} = \mathbb{B}_{2,NT}^{(2)},$$

both of which are of $O_P(h^{-1/2})$. Let $\xi_{it} = (\varepsilon_{it}, X_{it}')'$ and $\xi_i = (\xi_{i1}, ..., \xi_{iT})'$. Let $\varsigma_{ij,sr} = \frac{1}{T} \sum_{t=1}^{T} k_{h,st}^* k_{h,rt}^* \mathcal{X}_{is}^{(t)\prime} \mathbb{D}^{(t)} \mathcal{X}_{jr}^{(t)}$ and $W_{ij} \equiv W(\xi_i, \xi_j) = \frac{2h^{1/2}}{NT} \sum_{s,r=1}^{T} \varepsilon_{is} \varepsilon_{jr} \varsigma_{ij,sr}$. Then $A_{1,1}^{(b)} = \sum_{1 \le i < j \le N} W_{ij}$. Define

$$G_1 = \sum_{1 \le i < j \le N} E_{\mathcal{C}}(W_{ij}^4), \; G_2 = \sum_{1 \le i < j < k \le N} E_{\mathcal{C}}(W_{ij}^2 W_{ik}^2 + W_{ji}^2 W_{jk}^2 + W_{ki}^2 W_{kj}^2), \text{ and}$$

$$G_3 = \sum_{1 \le i < j < k < l \le N} E_{\mathcal{C}}(W_{ij} W_{ik} W_{lj} W_{lk} + W_{ij} W_{il} W_{kj} W_{kl} + W_{ik} W_{il} W_{jk} W_{jl}).$$

Let $\mathbb{V}_{NT}^{(2)} = \text{Var}_{\mathcal{C}}(A_{1,1}^{(b)})$. Note that $\mathbb{V}_{NT}^{(2)} = \sum_{1 \le i < j \le N} E_{\mathcal{C}}(W_{ij}^2) = \frac{4h}{N^2T^2} \sum_{1 \le i < j \le N} E_{\mathcal{C}} \left(\sum_{s \ne r} \varepsilon_{is} \varepsilon_{jr} \varsigma_{ij,sr} \right)^2 \xrightarrow{p} \mathbb{V}_0^{(2)} > 0$. By straightforward moment calculations and the repeated use of Lemma B.1, we can show that

$$G_1 = \frac{16h^2}{N^4T^4} \sum_{1 \le i < j \le N} E_{\mathcal{C}} \left(\sum_{s \ne r} \varepsilon_{is} \varepsilon_{jr} \varsigma_{ij,sr} \right)^4 = O_P\left(N^{-2}h^{-2}\right),$$

$$G_2 \lesssim \frac{16h^2}{N^4T^4} \sum_{i,j,k \text{ are all distinct}} E_{\mathcal{C}} \left[\left(\sum_{s \ne r} \varepsilon_{is} \varepsilon_{jr} \varsigma_{ij,sr} \right)^2 \left(\sum_{s \ne r} \varepsilon_{is} \varepsilon_{kr} \varsigma_{ik,sr} \right)^2 \right] = O_P\left(N^{-1}\right),$$

and similarly, $G_3 = O_P\left(h^2\right)$. Consequently, $G_l = o_P((\mathbb{V}_{NT}^{(2)})^2)$ for $l = 1, 2, 3$. Then by a conditional version of the CLT in Proposition 3.2 in de Jong (1987), we have $(\mathbb{V}_{NT}^{(2)})^{-1/2} A_{1,1}^{(b)} \xrightarrow{d} N(0,1)$ conditional on $\mathcal{C}$. The unconditional limiting law is also given by $N(0,1)$. Then $A_{1,1} - \mathbb{B}_{NT}^{(2)} \xrightarrow{d} N(0, \mathbb{V}_0^{(2)})$.

Under the additional bandwidth condition that $NTh^{9/2} = o(1)$, we have that $A_{1,2} = NTh^{1/2} o_P(a_{2NT}^2) = o_P(1)$. With some tedious calculations, we can show that $A_{1,3} = 2Nh^{1/2} \sum_{t=1}^{T} \left[D(F^{(t)})^{-1} \frac{1}{NT} \sum_{i=1}^{N} \sum_{s=1}^{T} \mathfrak{X}_{is}^{(t)} \varepsilon_{is}^{(t)} \right]' R_\beta(t) = o_P(1)$. Then $A_1 - \mathbb{B}_{NT}^{(2)} \xrightarrow{d} N(0, \mathbb{V}_0^{(2)})$.

Now, we consider the A_2 term under the local alternative $\mathbb{H}_A^{(2)}(a_{2NT})$ with $a_{2NT} = (NT)^{-1/2} h^{-1/4}$:

$$A_2 = Nh^{1/2} \sum_{t=1}^{T} \left\| \frac{1}{T} \sum_{s=1}^{T} \hat{\beta}_s^{bc} - \beta_t \right\|^2 = Nh^{1/2} \sum_{t=1}^{T} \left\| \frac{1}{T} \sum_{s=1}^{T} (\hat{\beta}_s^{bc} - \beta_s) - \frac{1}{T} \sum_{s=1}^{T} a_{2NT} \left(g_{0t} - g_{0s} \right) \right\|^2$$

$$= Nh^{1/2}\sum_{t=1}^{T}\left\|\frac{1}{T}\sum_{s=1}^{T}(\hat{\beta}_s^{bc}-\beta_s)\right\|^2 + Nh^{1/2}\sum_{t=1}^{T}\left\|\frac{1}{T}\sum_{s=1}^{T}a_{2NT}\left(g_{0t}-g_{0s}\right)\right\|^2$$
$$-2T^{-2}Nh^{1/2}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{r=1}^{T}a_{2NT}(\hat{\beta}_s^{bc}-\beta_s)'\left(g_{0t}-g_{0r}\right)\equiv A_{2,1}+A_{2,2}-2A_{2,3}.$$

Under $\mathbb{H}_A^{(2)}(a_{2NT})$, we can follow the proofs of Theorems 3.2 and 3.4 and show that $\frac{1}{T}\sum_{s=1}^{T}(\hat{\beta}_s^{bc}-\beta_s) = O_P((NT)^{-1/2}+h^2)$. Then $A_{2,1}=NTh^{1/2}O_P((NT)^{-1}+h^4)=O_P(h^{1/2}+NTh^{9/2})=o_P(1)$. For $A_{2,2}$, we have

$$A_{2,2}=\frac{1}{T^3}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{r=1}^{T}\left(g_{0t}-g_{0s}\right)'\left(g_{0t}-g_{0r}\right)=\frac{1}{T}\sum_{t=1}^{T}g_{0t}'g_{0t}+O(T^{-1})$$
$$=\int_0^1\left\|g_0(\tau)\right\|^2 d\tau+O_P(T^{-1})\equiv\Pi_2+O(T^{-1}),$$

where the second equality holds because $\frac{1}{T}\sum_{s=1}^{T}g_{0s}=\frac{1}{T}\sum_{s=1}^{T}g_0(\frac{s}{T})=\int_0^1 g_0\left(\tau\right)d\tau+O(\frac{1}{T})=O(\frac{1}{T})$ under our normalization restriction on $g_0\left(\cdot\right)$. By CS inequality, we have $|A_{2,3}|\leq\left\{A_{2,1}\right\}^{1/2}\left\{A_{2,2}\right\}^{1/2}=o_P(1)$. It follows that $A_2=\Pi_2+o_P(1)$.

Now, we show that $A_3=o_P(1)$. Noting that $\bar{\hat{\beta}}^{bc}-\beta_t=\frac{1}{T}\sum_{s=1}^{T}(\hat{\beta}_s^{bc}-\beta_s)-\frac{1}{T}\sum_{s=1}^{T}a_{2NT}\left(g_{0t}-g_{0s}\right)$, we can decompose A_3 as follows:

$$A_3=Nh^{1/2}\sum_{t=1}^{T}(\hat{\beta}_t^{bc}-\beta_t)'\frac{1}{T}\sum_{s=1}^{T}(\hat{\beta}_s^{bc}-\beta_s)-Nh^{1/2}\sum_{t=1}^{T}(\hat{\beta}_t^{bc}-\beta_t)'\frac{1}{T}\sum_{s=1}^{T}a_{2NT}\left(g_{0t}-g_s\right)\equiv A_{3,1}-A_{3,2}.$$

$A_{3,1}$ is the same as $A_{2,1}$, which is $o_P(1)$. For $A_{3,2}$, we have

$$\begin{aligned}A_{3,2} &= (NT)^{1/2}h^{1/4}\frac{1}{T}\sum_{t=1}^{T}(\hat{\beta}_t^{bc}-\beta_t)\frac{1}{T}\sum_{s=1}^{T}\left(g_{0t}-g_{0s}\right)\\ &= (NT)^{1/2}h^{1/4}\frac{1}{T}\sum_{t=1}^{T}(\hat{\beta}_t^{bc}-\beta_t)'g_{0t}-(NT)^{1/2}h^{1/4}\frac{1}{T}\sum_{t=1}^{T}(\hat{\beta}_t^{bc}-\beta_t)O(T^{-1}).\end{aligned}$$

Noting that g_{0t} is nonrandom, we can follow the proofs of Theorems 3.2 and 3.4 and show that $\frac{1}{T}\sum_{t=1}^{T}(\hat{\beta}_t^{bc}-\beta_t)'g_{0t}=O_P((NT)^{-1/2}+h^2)$. Then $A_{3,2}=O_P(h^{1/4}+(NT)^{1/2}h^{9/4})=o_P(1)$ and $A_3=o_P(1)$. In sum, we have shown that $TNh^{1/2}\hat{M}^{(2)}-\mathbb{B}_{NT}^{(2)}-\Pi_2\xrightarrow{d}N(0,\mathbb{V}_0^{(2)})$, where $\Pi_2=0$ under $\mathbb{H}_0^{(2)}$ and $\Pi_2=\int_0^1\left\|g_0(\tau)\right\|^2d\tau$ under $\mathbb{H}_A^{(2)}(a_{2NT})$.

(iii) As above, we show the asymptotic distribution of $TN^{1/2}h^{1/2}\hat{M}^{(3)}$ under $\mathbb{H}_A^{(3)}(a_{1NT})$ with $a_{1NT}=N^{-1/4}T^{-1/2}h^{-1/4}$. As in the proof of part (i), we can make the following decomposition:

$$\hat{\lambda}_{it}^{W\prime}\hat{F}_t^W-\tilde{\lambda}_{i0}^{W\prime}\tilde{F}_t^W=d_{1it}^W+d_{2it}^W+d_{3it}^W,$$

where for $l=1,2,3$, d_{lit}^W is defined analogously to d_{lit} with $\hat{\lambda}_{it}$, $\hat{F}_t$, $\tilde{\lambda}_{i0}$ and $\tilde{F}_t$ replaced by $\hat{\lambda}_{it}^W$, $\hat{F}_t^W$, $\tilde{\lambda}_{i0}^W$ and $\tilde{F}_t^W$, respectively. Given the fast convergence rate of $\hat{\beta}_t^{bc}-\beta_t$ under our bandwidth condition, the error in the estimation of β_t is asymptotically negligible so that we can readily show that under $\mathbb{H}_A^{(3)}(a_{1NT})$,

$$TN^{1/2}h^{1/2}\hat{M}^{(3)}=M_1+M_2+o_P\left(1\right),$$

where $M_1 - \mathbb{B}_{NT}^{(1)} - \Pi_1^{(1)} \xrightarrow{d} N(0, \mathbb{V}_0^{(3)})$, $M_2 = \Pi_1^{(2)} + o_P(1)$, and $\mathbb{V}_0^{(3)} = \mathbb{V}_0^{(1)}$. Consequently, we have

$$TN^{1/2}h^{1/2}\hat{M}^{(3)} - \mathbb{B}_{NT}^{(3)} - \Pi_3 \xrightarrow{d} N(0, \mathbb{V}_0^{(3)}),$$

where $\mathbb{B}_{NT}^{(3)} = \mathbb{B}_{NT}^{(1)}$ and $\Pi_3 = \Pi_1^{(1)} + \Pi_1^{(2)}$. This completes the proof of the theorem. ■

Proof of Theorem 4.2. (i) By Theorem 4.1(i), it suffices to show (i1) $\hat{\mathbb{B}}_{NT}^{(1)} - \mathbb{B}_{NT}^{(1)} = o_p(1)$ and (i2) $\hat{\mathbb{V}}_{NT}^{(1)} - \mathbb{V}_{NT}^{(1)} = o_p(1)$. For the use in the proof of Theorem 4.3(i), we prove the claims under $\mathbb{H}_1^{(1)}(a_{1NT})$. Recall that

$$\begin{aligned}
\hat{L}_{st} &= k_{h,st}^{*}\hat{F}_s'\hat{F}_t - \tilde{F}_s'\tilde{F}_t,\ \hat{\Gamma}_{it,j} = T^{-1}\sum_{s=j+1}^{T}\hat{L}_{st}\hat{L}_{s-j,t}\varepsilon_{is}\varepsilon_{i,s-j},\ \text{and}\ \hat{\Xi}_{it} = \hat{\Gamma}_{it,0} + 2\sum_{j=1}^{l_T} w_{Tj}\hat{\Gamma}_{it,j}.\\
\hat{\mathbb{B}}_{NT}^{(1)} &= \frac{h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\hat{\eta}_{it,s}^2 + \frac{2h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{j=1}^{l_T} w_{Tj}\sum_{s=j+1}^{T}\hat{\eta}_{it,s}\hat{\eta}_{it,s-j}.
\end{aligned}$$

Let $\eta_{it,s} = F_t' L_{st} F_s \varepsilon_{is}$ and $\hat{\eta}_{it,s} = (k_{h,st}^{*}\hat{F}_s'\hat{F}_t - \tilde{F}_s'\tilde{F}_t)\hat{\varepsilon}_{is}$ and $\bar{\eta}_{it,s} = (k_{h,st}^{*}\hat{F}_s'\hat{F}_t - \tilde{F}_s'\tilde{F}_t)\varepsilon_{is}$. Then

$$\begin{aligned}
\mathbb{B}_{NT}^{(1)} &= \frac{h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T} E_{\mathcal{C}}\left(\sum_{s=1}^{T}\eta_{it,s}\right)^2\\
&= \frac{h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T} E_{\mathcal{C}}(\eta_{it,s}^2) + \frac{2h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{j=1}^{T-1}\sum_{s=j+1}^{T} E_{\mathcal{C}}(\eta_{it,s}\eta_{it,s-j}).
\end{aligned}$$

Let $\bar{\mathbb{B}}_{NT}^{(1)} = \frac{h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\eta_{it,s}^2 + \frac{2h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{j=1}^{l_T} w_{Tj}\sum_{s=j+1}^{T}\eta_{it,s}\eta_{it,s-j}$. Then

$$\begin{aligned}
\hat{\mathbb{B}}_{NT}^{(1)} - \mathbb{B}_{NT}^{(1)} &= \left[\hat{\mathbb{B}}_{NT}^{(1)} - \bar{\mathbb{B}}_{NT}^{(1)}\right] + \left[\bar{\mathbb{B}}_{NT}^{(1)} - \mathbb{B}_{NT}^{(1)}\right]\\
&= \left[\hat{\mathbb{B}}_{NT}^{(1)} - \bar{\mathbb{B}}_{NT}^{(1)}\right] + \frac{h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}[\eta_{it,s}^2 - E_{\mathcal{C}}(\eta_{it,s}^2)]\\
&\quad + \frac{h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{j=1}^{l_T} w_{Tj}\sum_{s=j+1}^{T}[\eta_{it,s}\eta_{it,s-j} - E_{\mathcal{C}}(\eta_{it,s}\eta_{it,s-j})]\\
&\quad + \left[\frac{h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\left\{\sum_{s=1}^{T}[E_{\mathcal{C}}(\eta_{it,s}^2) + 2\sum_{j=1}^{l_T} w_{Tj}\sum_{s=j+1}^{T} E_{\mathcal{C}}(\eta_{it,s}\eta_{it,s-j})\right\} - \mathbb{B}_{NT}^{(1)}\right] \equiv \sum_{\ell=1}^{4} B_\ell.
\end{aligned}$$

It suffices to show that $B_l = o_P(1)$ for $l \in [4]$.

We first show that $B_1 = o_P(1)$. Note that

$$\begin{aligned}
B_1 &= \hat{\mathbb{B}}_{NT}^{(1)} - \bar{\mathbb{B}}_{NT}^{(1)}\\
&= \frac{h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(\hat{\eta}_{it,s}^2 - \eta_{it,s}^2\right) + \frac{2h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{j=1}^{l_T} w_{Tj}\sum_{s=j+1}^{T}\left(\hat{\eta}_{it,s}\hat{\eta}_{it,s-j} - \eta_{it,s}\eta_{it,s-j}\right)\\
&\equiv B_{1,1} + 2B_{1,2}.
\end{aligned}$$

We will show that $B_1 = o_P(1)$ by showing that $B_{1,1} = o_P\left(l_T^{-1}\right)$ and $B_{1,2} = o_P(1)$. Using $a^2 - b^2 = (a-b)^2 + 2(a-b)b$, we obtain

$$B_{1,1} = \frac{h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(\hat{\eta}_{it,s}^2 - \bar{\eta}_{it,s}^2\right) + \frac{h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(\bar{\eta}_{it,s}^2 - \eta_{it,s}^2\right)$$

$$\begin{aligned}
&= \frac{h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(\hat{\eta}_{it,s}-\bar{\eta}_{it,s}\right)^2+\frac{2h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(\hat{\eta}_{it,s}-\bar{\eta}_{it,s}\right)\bar{\eta}_{it,s}\\
&\quad+\frac{h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(\bar{\eta}_{it,s}-\eta_{it,s}\right)^2+\frac{2h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(\bar{\eta}_{it,s}-\eta_{it,s}\right)\eta_{it,s}\\
&\equiv B_{1,11}+2B_{1,12}+B_{1,13}+2B_{1,14}.
\end{aligned}$$

Using $\hat{\eta}_{it,s}-\bar{\eta}_{it,s}=(k^*_{h,st}\hat{F}_s'\hat{F}_t-\tilde{F}_s'\tilde{F}_t)\left(\hat{\varepsilon}_{is}-\varepsilon_{is}\right)$, $\bar{\eta}_{it,s}-\eta_{it,s}=[(k^*_{h,st}\hat{F}_s'\hat{F}_t-\tilde{F}_s'\tilde{F}_t)-F_t'L_{st}F_s]\varepsilon_{is}$, the expansion for $\hat{\varepsilon}_{is}-\varepsilon_{is}$, and the mean square convergence of $\hat{F}_t$ and $\tilde{F}_t$ under the local alternatives, one can readily show that $B_{1,1l}=o_P\left(l_T^{-1}\right)$ for $l\in[4]$. For example, for $B_{1,13}$, noting that

$$\begin{aligned}
\bar{\eta}_{it,s}-\eta_{it,s} &= \left[(k^*_{h,st}\hat{F}_s'\hat{F}_t-\tilde{F}_s'\tilde{F}_t)-F_t'\bar{L}_{st}F_s\right]\varepsilon_{is}\\
&= \left[(k^*_{h,st}\hat{F}_s'\hat{F}_t-\tilde{F}_s'\tilde{F}_t)-F_t'L_{st}F_s\right]\varepsilon_{is}+F_t'(L_{st}-\bar{L}_{st})F_s\varepsilon_{is}\\
&= k^*_{h,st}\left(\hat{F}_s'\hat{F}_t-F_s'H^{(t)}H^{(t)\prime}F_t\right)\varepsilon_{is}-(\tilde{F}_s'\tilde{F}_t-F_s'HH'F_t)\varepsilon_{is}\\
&\quad+k^*_{h,st}F_t'\left(H^{(t)}H^{(t)\prime}-H_0H_0'\right)F_s\varepsilon_{is}-F_t'\left(HH'-H_0H_0'\right)F_s\varepsilon_{is}\equiv\sum_{l=1}^{4}\eta_{it,s}\left(l\right),
\end{aligned}$$

we have

$$B_{1,13}=\frac{h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(\bar{\eta}_{it,s}-\eta_{it,s}\right)^2\leq 4\sum_{l=1}^{4}\frac{h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\left[\eta_{it,s}\left(l\right)\right]^2\equiv 4\sum_{l=1}^{4}B_{1,13}\left(l\right).$$

Noting that $\hat{F}_t'\hat{F}_s-F_t'H^{(t)}H^{(t)\prime}F_s=(\hat{F}_t-H^{(t)\prime}F_t)'(\hat{F}_s-H^{(t)\prime}F_s)+(\hat{F}_t-H^{(t)\prime}F_t)'H^{(t)\prime}F_s+F_t'H^{(t)}(\hat{F}_s-H^{(t)\prime}F_s)\equiv\sum_{l=1}^{3}\varsigma_{l,ts}$

$$\begin{aligned}
B_{1,13}\left(1\right) &= \frac{h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}k^{*2}_{h,st}[\hat{F}_s'\hat{F}_t-F_s'H^{(t)}H^{(t)\prime}F_t]^2\varepsilon_{is}^2\leq 3\sum_{l=1}^{3}\frac{h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}k^{*2}_{h,st}\varsigma^2_{st,l}\varepsilon^2_{is}\\
&\equiv \sum_{l=1}^{3}\varsigma_l.
\end{aligned}$$

Using the fact that $H^{(t)}=H+O_P\left(C_{0NT}^{-1}\right)$ under $\mathbb{H}_1^{(1)}(a_{1NT})$ and $\max_t\frac{1}{T}\sum_{s=1}^{T}k^{*2}_{h,st}\left\|\hat{F}_s-H^{(t)\prime}F_s\right\|^2=O\left(h^{-1}C_{NT}^{-2}\ln N\right)$, we can readily show that $\varsigma_1=N^{1/2}h^{1/2}O_P\left(h^{-1}C_{NT}^{-4}\ln N\right)=o_P\left(l_T^{-1}\right)$, and $\varsigma_l=N^{1/2}h^{1/2}$ $\times O_P\left(h^{-1}C_{NT}^{-2}\ln N\right)=o_P\left(l_T^{-1}\right)$ for $l=2,3$. Then $B_{1,13}\left(1\right)=o_P\left(l_T^{-1}\right)$. By the same token, $B_{1,13}\left(2\right)=o_P\left(l_T^{-1}\right)$. For $B_{1,13}\left(3\right)$ and $B_{1,13}\left(4\right)$, we have

$$B_{1,13}\left(3\right)=\frac{h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}k^{*2}_{h,st}[F_t'(H^{(t)}H^{(t)\prime}-H_0H_0')F_s]^2\varepsilon_{is}^2=N^{1/2}h^{1/2}O_P\left(h^{-1}C_{NT}^{-2}\ln N\right)=o_P\left(l_T^{-1}\right)$$

and

$$B_{1,13}\left(4\right)=\frac{h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}[F_t'\left(HH'-H_0H_0'\right)F_s]^2\varepsilon_{is}^2=N^{1/2}h^{1/2}O_P\left(C_{0NT}^{-2}\right)=o_P\left(l_T^{-1}\right).$$

Consequently, $B_{1,13}=o_P\left(l_T^{-1}\right)$. It follows that $B_{1,1}=o_P\left(l_T^{-1}\right)$.

By the uniform boundedness of w_{Tj} and arguments similar to those used in the analysis of $B_{1,1}$, we can show that $B_{1,2}=l_To_P\left(l_T^{-1}\right)=o_P\left(1\right)$. Then $B_1=o_P\left(1\right)$.

Next, we consider B_2 :

$$B_2 = \frac{h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}[\eta_{it,s}^2 - E_{\mathcal{C}}(\eta_{it,s}^2)] = \frac{h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}(k_{h,st}^* - 1)^2\left(F_t'F_s\right)^2\left[\varepsilon_{is}^2 - E_{\mathcal{C}}(\varepsilon_{is}^2)\right].$$

Note that $E_{\mathcal{C}}\left(B_2\right) = 0$ and

$$\begin{aligned}
\mathrm{Var}_{\mathcal{C}}\left(B_2\right) &= \frac{h}{NT^4}\sum_{i=1}^{N}\mathrm{Var}_{\mathcal{C}}\left(\sum_{t=1}^{T}\sum_{s=1}^{T}(k_{h,st}^* - 1)^2\left(F_t'F_s\right)^2\left[\varepsilon_{is}^2 - E_{\mathcal{C}}(\varepsilon_{is}^2)\right]\right)\\
&\leq \frac{h}{NT^4}\sum_{i=1}^{N}\mathrm{Var}_{\mathcal{C}}\left(\sum_{s=1}^{T}(k_{h,ss}^* - 1)^2\left(F_s'F_s\right)^2\left[\varepsilon_{is}^2 - E_{\mathcal{C}}(\varepsilon_{is}^2)\right]\right)\\
&\quad + \frac{4h}{NT^4}\sum_{i=1}^{N}\mathrm{Var}_{\mathcal{C}}\left(\sum_{1\leq t<s\leq T}(k_{h,st}^* - 1)^2\left(F_t'F_s\right)^2\left[\varepsilon_{is}^2 - E_{\mathcal{C}}(\varepsilon_{is}^2)\right]\right).
\end{aligned}$$

For the first term, we can readily obtain a rough bound $O_P(T^{-2}h^{-3}) = o_P\left(1\right);$ for the second term, an application of Lemma B.1 yields the probability order $O_P\left(T^{-2}h^{-1} + T^{-1}h\right) = o_P\left(1\right)$. Thus $B_2 = o_P\left(1\right)$.

Next, we consider B_3 :

$$B_3 = \frac{h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{j=1}^{l_T}w_{Tj}\sum_{s=j+1}^{T}\left[\eta_{it,s}\eta_{it,s-j} - E_{\mathcal{C}}(\eta_{it,s}\eta_{it,s-j})\right] = \sum_{j=1}^{l_T}w_{Tj}B_{3,j},$$

where $B_{3,j} = \frac{h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=j+1}^{T}(k_{h,st}^* - 1)(k_{h,s-j,t}^* - 1)F_t'F_sF_t'F_{s-j}\left[\varepsilon_{is}\varepsilon_{i,s-j} - E_{\mathcal{C}}(\varepsilon_{is}\varepsilon_{i,s-j})\right]$. By moment calculations, we can show that $E[B_{3,j}^2] = E\{E_{\mathcal{C}}[B_{3,j}^2]\} \leq CT^{-1}$ for some constant C. Then by the union bound and Chebyshev inequality, we have that for any $\epsilon > 0$

$$\begin{aligned}
P\left(|B_3| \geq \epsilon\right) &= P\left(\left|\sum_{j=1}^{l_T}w_{Tj}B_{3,j}\right| \geq \epsilon\right) \leq P\left(\sum_{j=1}^{l_T}|w_{Tj}|\,|B_{3,j}| \geq \epsilon\right) \leq P\left(\sum_{j=1}^{l_T}|B_{3,j}| \geq \epsilon/c_w\right)\\
&\leq \sum_{j=1}^{l_T}P\left(\sum_{j=1}^{l_T}|B_{3,j}| \geq \epsilon/(c_wl_T)\right) \leq \frac{(c_wl_T)^2}{\epsilon^2}\sum_{j=1}^{l_T}E[B_{3,j}^2] = O(l_T^3T^{-1}) = o(1).
\end{aligned}$$

Then $B_3 = o_P\left(1\right)$.

For B_4, we have

$$\begin{aligned}
B_4 &= \frac{2h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\left[\sum_{j=1}^{l_T}w_{Tj}\sum_{s=j+1}^{T}E_{\mathcal{C}}(\eta_{it,s}\eta_{it,s-j}) - \sum_{j=1}^{T-1}\sum_{s=j+1}^{T}E_{\mathcal{C}}(\eta_{it,s}\eta_{it,s-j})\right]\\
&= \frac{2h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{j=1}^{l_T}(w_{Tj} - 1)\sum_{s=j+1}^{T}E_{\mathcal{C}}(\eta_{it,s}\eta_{it,s-j}) + \frac{2h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{j=l_T+1}^{T-1}\sum_{s=j+1}^{T}E_{\mathcal{C}}(\eta_{it,s}\eta_{it,s-j})\\
&\equiv 2B_{4,1} + 2B_{4,2}.
\end{aligned}$$

For $B_{4,2}$, we have

$$E|B_{4,2}| = \frac{2h^{1/2}}{N^{1/2}T^2}E\left|\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{j=l_T+1}^{T-1}\sum_{s=j+1}^{T}(k_{h,st}^* - 1)(k_{h,s-j,t}^* - 1)F_t'F_sF_t'F_{s-j}E_{\mathcal{C}}(\varepsilon_{is}\varepsilon_{i,s-j})\right|$$

$$\begin{aligned}
&\lesssim \frac{h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{j=l_T+1}^{T-1} E\left|\sum_{s=j+1}^{T}(k^*_{h,st}-1)(k^*_{h,s-j,t}-1)F_t'F_sF_t'F_{s-j}\alpha^{\frac{3+2\eta}{4+2\eta}}(j)\right| \\
&\lesssim \frac{h^{-1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{j=l_T+1}^{T-1}\sum_{t=j+1}^{T}\alpha^{\frac{3+2\eta}{4+2\eta}}(j)+\frac{h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{j=l_T+1}^{T-1}\sum_{s=j+1,s\neq t}^{T}\alpha^{\frac{3+2\eta}{4+2\eta}}(j) \\
&\lesssim \frac{N^{1/2}h^{-1/2}}{Tl_T^a}\sum_{j=l_T+1}^{T} j^{a_0}\alpha^{\frac{3+2\eta}{4+2\eta}}(j)+\frac{N^{1/2}h^{1/2}}{l_T^a}\sum_{j=l_T+1}^{T} j^{a_0}\alpha^{\frac{3+2\eta}{4+2\eta}}(j)=o(1).
\end{aligned}$$

For $B_{4,1}$, we have

$$|B_{4,1}|\leq \frac{h^{1/2}}{N^{1/2}T^2}\sum_{j=1}^{l_T}|w_{Tj}-1|\left|\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=j+1}^{T}E_{\mathcal{C}}(\eta_{it,s}\eta_{it,s-j})\right|,$$

where the right hand side is dominated by $\bar{B}_{4,1}\equiv \frac{h^{1/2}}{N^{1/2}T^2}\sum_{j=1}^{l_T}|w_{Tj}-1|\left|\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=j+1}^{T}E_{\mathcal{C}}(\eta_{it,s}\eta_{it,s-j})\right|$.

$$\begin{aligned}
\bar{B}_{4,1} &= \frac{h^{1/2}}{N^{1/2}T^2}\sum_{j=1}^{l_T}\left|\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=j+1}^{T}E_{\mathcal{C}}(\eta_{it,s}\eta_{it,s-j})\right| \\
&= \frac{h^{1/2}}{N^{1/2}T^2}\sum_{j=1}^{l_T}\left|\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=j+1}^{T}(k^*_{h,st}-1)(k^*_{h,s-j,t}-1)F_t'F_sF_t'F_{s-j}E_{\mathcal{C}}(\varepsilon_{is}\varepsilon_{i,s-j})\right| \\
&= \frac{h^{1/2}}{N^{1/2}T^2}\sum_{t=1}^{l_T}\left|\sum_{i=1}^{N}\sum_{s=t+1}^{T}\left|(k^*_{h,st}-1)(k^*_{h,s-t,t}-1)\right|F_t'F_sF_t'F_{s-t}E_{\mathcal{C}}(\varepsilon_{is}\varepsilon_{i,s-t})\right| \\
&\quad+\frac{h^{1/2}}{N^{1/2}T^2}\sum_{j=1}^{l_T}\left|\sum_{i=1}^{N}\sum_{s=j+1}^{T}\left|(k^*_{h,ss}-1)(k^*_{h,s-j,s}-1)\right|F_t'F_sF_t'F_{s-j}E_{\mathcal{C}}(\varepsilon_{is}\varepsilon_{i,s-j})\right| \\
&\quad+\frac{h^{1/2}}{N^{1/2}T^2}\sum_{j=1}^{l_T}\left|\sum_{i=1}^{N}\sum_{s=j+1}^{T}\sum_{t=1,t\neq s,j}^{T}\left|(k^*_{h,st}-1)(k^*_{h,s-j,t}-1)\right|F_t'F_sF_t'F_{s-j}E_{\mathcal{C}}(\varepsilon_{is}\varepsilon_{i,s-j})\right|\equiv\sum_{\ell=1}^{3}\bar{B}_{4,1\ell}.
\end{aligned}$$

By Lemma B.1 with $\tilde{\eta}=3+2\eta$, we can readily show that

$$\begin{aligned}
E\left[\bar{B}_{4,11}\right] &= \frac{h^{1/2}}{N^{1/2}T^2}\sum_{t=1}^{l_T}\sum_{i=1}^{N}\sum_{s=t+1}^{T}\left|(k^*_{h,st}-1)(k^*_{h,s-t,t}-1)\right|E\left[F_t'F_sF_t'F_{s-t}E_{\mathcal{C}}(\varepsilon_{is}\varepsilon_{i,s-t})\right] \\
&\lesssim \frac{h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{t=1}^{l_T}\sum_{s=t+1}^{T}\left|(k^*_{h,st}-1)(k^*_{h,s-t,t}-1)\right|\alpha^{\frac{3+2\eta}{4+2\eta}}(t)=O(N^{1/2}T^{-1}h^{-1/2})=o(1).
\end{aligned}$$

Then $\bar{B}_{4,11}=o_P(1)$. Similarly, $\bar{B}_{4,12}=o_P(1)$. For $\bar{B}_{4,13}$, we have

$$\begin{aligned}
\bar{B}_{4,13} &= \frac{h^{1/2}}{N^{1/2}T^2}\sum_{j=1}^{l_T}\sum_{i=1}^{N}\sum_{s=j+1}^{T}\sum_{t=1,t\neq s,j}^{T}\left|(k^*_{h,st}-1)(k^*_{h,s-j,t}-1)\right|\left|F_t'F_sF_t'F_{s-j}E_{\mathcal{C}}(\varepsilon_{is}\varepsilon_{i,s-j})\right| \\
&\lesssim \frac{h^{1/2}}{N^{1/2}T^2}\sum_{i=1}^{N}\sum_{r=1}^{T-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=r+1}^{(r+l_T)\wedge T}\left|(k^*_{h,st}-1)(k^*_{h,rt}-1)\right|\|F_t\|^2\left(\|F_s\|^2+\|F_r\|^2\right)|E_{\mathcal{C}}(\varepsilon_{is}\varepsilon_{ir})| \\
&= O_P(N^{1/2}T^{-1}h^{-1/2}l_T)=o_P(1).
\end{aligned}$$

Then $\bar{B}_{4,1} = o_P(1)$. Since $\lim_{T\to\infty} w_{Tj} = 1$ for each j, it follows from the dominated convergence theorem that $B_{4,1} = o_P(1)$.

In sum, we have shown that $\hat{\mathbb{B}}_{NT}^{(1)} - \mathbb{B}_{NT}^{(1)} = o_p(1)$.

Now, we prove $\hat{\mathbb{V}}_{NT}^{(1)} - \mathbb{V}_{NT}^{(1)} = o_p(1)$. Let $\bar{\mathbb{V}}_{NT}^{(1)} = \frac{h}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{s,r=1}^{T}\xi_{h,sr}\varepsilon_{is}\varepsilon_{ir}\right)^2$.

$$
\begin{aligned}
&\hat{\mathbb{V}}_{NT}^{(1)} - \mathbb{V}_{NT}^{(1)} \\
=\ & \hat{\mathbb{V}}_{NT}^{(1)} - \bar{\mathbb{V}}_{NT}^{(1)} + (\bar{\mathbb{V}}_{NT}^{(1)} - \mathbb{V}_{NT}^{(1)}) \\
=\ & \frac{h}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{s,r=1}^{T}\hat{\xi}_{h,sr}\hat{\varepsilon}_{ir}\hat{\varepsilon}_{is}\right)^2 - \frac{h}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{s,r=1}^{T}\xi_{h,sr}\varepsilon_{is}\varepsilon_{ir}\right)^2 + (\bar{\mathbb{V}}_{NT}^{(1)} - \mathbb{V}_{NT}^{(1)}) \\
=\ & \frac{h}{N}\sum_{i=1}^{N}\left[\frac{1}{T}\sum_{s,r=1}^{T}(\hat{\xi}_{h,sr}\hat{\varepsilon}_{ir}\hat{\varepsilon}_{is} - \xi_{h,sr}\varepsilon_{is}\varepsilon_{ir})\right]^2 + \frac{2h}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{s,r=1}^{T}(\hat{\xi}_{h,sr}\hat{\varepsilon}_{ir}\hat{\varepsilon}_{is} - \xi_{h,sr}\varepsilon_{is}\varepsilon_{ir})\right)\left(\frac{1}{T}\sum_{s,r=1}^{T}\xi_{h,sr}\varepsilon_{is}\varepsilon_{ir}\right) \\
& +(\bar{\mathbb{V}}_{NT}^{(1)} - \mathbb{V}_{NT}^{(1)}) \equiv V_1 + 2V_2 + V_3,
\end{aligned}
$$

where $\hat{\xi}_{h,sr} \equiv \frac{1}{T}\sum_{t=1}^{T} k_{h,st}^{*}k_{h,rt}^{*}\tilde{F}_r'\hat{F}_t\hat{F}_t'\tilde{F}_s$ and $\xi_{h,sr} \equiv \frac{1}{T}\sum_{t=1}^{T} k_{h,st}^{*}k_{h,rt}^{*}F_r'H_0H_0'F_tF_t'H_0H_0'F_s$. Note that

$$
\begin{aligned}
\hat{\xi}_{h,sr}\hat{\varepsilon}_{ir}\hat{\varepsilon}_{is} - \xi_{h,sr}\varepsilon_{is}\varepsilon_{ir} &= (\hat{\xi}_{h,sr} - \xi_{h,sr})(\hat{\varepsilon}_{ir}\hat{\varepsilon}_{is} - \varepsilon_{is}\varepsilon_{ir}) + (\hat{\xi}_{h,sr} - \xi_{h,sr})\varepsilon_{is}\varepsilon_{ir} + \xi_{h,sr}(\hat{\varepsilon}_{ir}\hat{\varepsilon}_{is} - \varepsilon_{is}\varepsilon_{ir}) \\
&= (\hat{\xi}_{h,sr} - \xi_{h,sr})(\hat{\varepsilon}_{ir} - \varepsilon_{ir})(\hat{\varepsilon}_{is} - \varepsilon_{is}) + (\hat{\xi}_{h,sr} - \xi_{h,sr})(\hat{\varepsilon}_{ir} - \varepsilon_{ir})\varepsilon_{is} \\
&\quad +(\hat{\xi}_{h,sr} - \xi_{h,sr})(\hat{\varepsilon}_{is} - \varepsilon_{is})\varepsilon_{ir} + (\hat{\xi}_{h,sr} - \xi_{h,sr})\varepsilon_{is}\varepsilon_{ir} \\
&\quad +\xi_{h,sr}(\hat{\varepsilon}_{ir} - \varepsilon_{ir})(\hat{\varepsilon}_{is} - \varepsilon_{is}) + \xi_{h,sr}(\hat{\varepsilon}_{ir} - \varepsilon_{ir})\varepsilon_{is} + \xi_{h,sr}(\hat{\varepsilon}_{is} - \varepsilon_{is})\varepsilon_{ir} \\
&\equiv \sum_{l=1}^{7}\varphi_{l,isr}, \qquad \text{(B.4)}
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{\xi}_{h,sr} - \xi_{h,sr} &= \frac{1}{T}\sum_{t=1}^{T}k_{h,st}^{*}k_{h,rt}^{*}\left(\tilde{F}_r'\hat{F}_t\hat{F}_t'\tilde{F}_s - F_r'H_0H_0'F_tF_t'H_0H_0'F_s\right) \\
&= \frac{1}{T}\sum_{t=1}^{T}k_{h,st}^{*}k_{h,rt}^{*}\{(\tilde{F}_r - F_rH_0)'(\hat{F}_t\hat{F}_t' - H_0'F_tF_t'H_0)\tilde{F}_s + (\tilde{F}_r - F_rH_0)'H_0'F_tF_t'H_0(\tilde{F}_s - H_0'F_s) \\
&\quad +F_r'H_0(\hat{F}_t\hat{F}_t' - H_0'F_tF_t'H_0)(\tilde{F}_s - H_0'F_s) + (\tilde{F}_r - F_rH_0)'H_0'F_tF_t'H_0H_0'F_s \\
&\quad +F_r'H_0(\hat{F}_t\hat{F}_t' - H_0'F_tF_t'H_0)H_0'F_s + F_r'H_0H_0'F_tF_t'H_0(\tilde{F}_s - H_0'F_s)\} \\
&\equiv \sum_{l=1}^{6}\xi_{lh,sr}. \qquad \text{(B.5)}
\end{aligned}
$$

In addition,

$$
\begin{aligned}
&\varepsilon_{is} - \hat{\varepsilon}_{is} \\
=\ & X_{is}'(\hat{\beta}_s^{bc} - \beta_s) + (\hat{\lambda}_{is}'\hat{F}_s - \lambda_{is}'F_s) \\
=\ & X_{is}'(\hat{\beta}_s^{bc} - \beta_s) + (\hat{\lambda}_{is} - H^{(s)-1}\lambda_{is})'H^{(s)\prime}F_s + \lambda_{is}'(H^{(s)-1})'(\hat{F}_s - H^{(s)\prime}F_s) + (\hat{\lambda}_{is} - H^{(s)-1}\lambda_{is})'(\hat{F}_s - H^{(s)\prime}F_s) \\
\equiv\ & \sum_{l=1}^{4}\varkappa_{l,is}.
\end{aligned}
$$

We can readily show that

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{s=1}^{T}(\hat{\varepsilon}_{is}-\varepsilon_{is})^2 \leq \sum_{l=1}^{4}\frac{1}{NT}\sum_{i=1}^{N}\sum_{s=1}^{T}\varkappa_{lis}^2 = O_P\left(C_{NT}^{-2}\ln T\right)$$

where, e.g., $\frac{1}{NT}\sum_{i=1}^{N}\sum_{s=1}^{T}\varkappa_{2is}^2 \lesssim \max_s \frac{1}{N}\sum_{i=1}^{N}\left\|\hat{\lambda}_{is}-H^{(s)-1}\lambda_{is}\right\|^2 \frac{1}{T}\sum_{s=1}^{T}\|F_s\|^2 = O_P\left(C_{NT}^{-2}\ln T\right)$. Similarly, we can show that

$$\begin{aligned}
\max_t \frac{1}{NT}\sum_{i=1}^{N}\sum_{s=1}^{T}k_{h,st}^*(\hat{\varepsilon}_{is}-\varepsilon_{is})^2 &\leq \max_t \sum_{l=1}^{4}\frac{1}{NT}\sum_{i=1}^{N}\sum_{s=1}^{T}k_{h,st}^*\varkappa_{lis}^2 = O_P\left(C_{NT}^{-2}\ln T\right),\\
\max_t \frac{1}{NT}\sum_{i=1}^{N}\sum_{s=1}^{T}k_{h,st}^*F_s(\hat{\varepsilon}_{is}-\varepsilon_{is}) &= \max_t \sum_{l=1}^{4}\frac{1}{NT}\sum_{i=1}^{N}\sum_{s=1}^{T}k_{h,st}^*F_s\varkappa_{lis} = O_P\left(C_{NT}^{-2}\ln T\right),\\
\frac{1}{N}\sum_{i=1}^{N}\left[\frac{1}{T}\sum_{r=1}^{T}(\hat{\varepsilon}_{ir}-\varepsilon_{ir})^2\right]^2 &= O_P\left(C_{NT}^{-4}(\ln T)^2\right) \text{ and } \frac{1}{T^2}\sum_{s,r=1}^{T}(\hat{\xi}_{h,sr}-\xi_{h,sr})^2 = O_P\left(C_{NT}^{-2}\right).
\end{aligned}$$

where, e.g., $\frac{1}{NT}\sum_{i=1}^{N}\sum_{s=1}^{T}k_{h,st}^*\varkappa_{2is}^2 \lesssim \max_s \frac{1}{N}\sum_{i=1}^{N}\left\|\hat{\lambda}_{is}-H^{(s)-1}\lambda_{is}\right\|^2 \max_t \frac{1}{T}\sum_{s=1}^{T}k_{h,st}^*\|F_s\|^2 = O_P\left(C_{NT}^{-2}\ln T\right)$ and $\frac{1}{NT}\sum_{i=1}^{N}\sum_{s=1}^{T}k_{h,st}^*F_s\varkappa_{2is} \lesssim \max_s \left|\frac{1}{N}\sum_{i=1}^{N}(\hat{\lambda}_{is}-H^{(s)-1}\lambda_{is})\right| \max_t \frac{1}{T}\sum_{s=1}^{T}k_{h,st}^*\|F_s\|^2 = O_P\left(C_{NT}^{-2}\ln T\right)O_P(1)$.

For V_1, we have

$$V_1 = \frac{h}{N}\sum_{i=1}^{N}\left[\frac{1}{T}\sum_{s,r=1}^{T}(\hat{\xi}_{h,sr}\hat{\varepsilon}_{ir}\hat{\varepsilon}_{is}-\xi_{h,sr}\varepsilon_{is}\varepsilon_{ir})\right]^2 \leq 7\sum_{l=1}^{7}\frac{h}{N}\sum_{i=1}^{N}\left[\frac{1}{T}\sum_{s,r=1}^{T}\varphi_{l,isr}\right]^2 \equiv \sum_{l=1}^{7}V_{1l}.$$

Noting that $V_{13}=V_{12}$ and $V_{17}=V_{16}$, it suffices to study V_{1l} for $l=1,2,4,5,6$. For V_{11}, it suffices to consider a rough bound:

$$\begin{aligned}
V_{11} &= \frac{h}{N}\sum_{i=1}^{N}\left[\frac{1}{T}\sum_{s,r=1}^{T}(\hat{\xi}_{h,sr}-\xi_{h,sr})(\hat{\varepsilon}_{ir}-\varepsilon_{ir})(\hat{\varepsilon}_{is}-\varepsilon_{is})\right]^2\\
&\leq T^2h\frac{1}{T^2}\sum_{s,r=1}^{T}(\hat{\xi}_{h,sr}-\xi_{h,sr})^2\frac{1}{N}\sum_{i=1}^{N}\left[\frac{1}{T}\sum_{r=1}^{T}(\hat{\varepsilon}_{ir}-\varepsilon_{ir})^2\right]^2 = T^2hO_P\left(C_{NT}^{-2}\right)O_P\left(C_{NT}^{-4}(\ln T)^2\right) = o_P(1).
\end{aligned}$$

To study V_{1l} for $l=2,4,5,6$, we first notice that

$$\max_{i,r}\left|\frac{1}{T}\sum_{s=1}^{T}(\hat{\xi}_{h,sr}-\xi_{h,sr})\varepsilon_{is}\right| \leq \sum_{l=1}^{6}\max_{i,r}\left|\frac{1}{T}\sum_{s=1}^{T}\xi_{hl,sr}\varepsilon_{is}\right| \equiv \sum_{l=1}^{6}I_l, \text{ say.}$$

We focus on the study of I_6 as the other terms are of the same or smaller order:

$$\begin{aligned}
I_6 &= \max_{i,r}\left|\frac{1}{T}\sum_{t=1}^{T}k_{h,rt}^*F_r'H_0H_0'F_tF_t'H_0\frac{1}{T}\sum_{s=1}^{T}k_{h,st}^*(\tilde{F}_s-H_0'F_s)\varepsilon_{is}\right|\\
&\lesssim \max_r\|F_r\|\max_r\frac{1}{T}\sum_{t=1}^{T}k_{h,rt}^*\|F_t\|^2\left\|\max_{i,t}\frac{1}{T}\sum_{s=1}^{T}k_{h,st}^*(\tilde{F}_s-H_0'F_s)\varepsilon_{is}\right\| = T^{1/(8+4\eta)}O_P\left(C_{NT}^{-2}\ln T\right).
\end{aligned}$$

Then $\max_{i,r}\left|\frac{1}{T}\sum_{s=1}^{T}(\hat{\xi}_{h,sr}-\xi_{h,sr})\varepsilon_{is}\right| = O_P\left(T^{1/(8+4\eta)}C_{NT}^{-2}\ln T\right)$. Similarly, we have

$$\begin{aligned}
\max_{i,r}\left|\frac{1}{T}\sum_{s=1}^{T}\xi_{h,sr}(\hat{\varepsilon}_{is}-\varepsilon_{is})\right| &= \max_{i,r}\left|\frac{1}{T}\sum_{s=1}^{T}\frac{1}{T}\sum_{t=1}^{T}k_{h,st}^{*}k_{h,rt}^{*}F_r'H_0H_0'F_tF_t'H_0H_0'F_s(\hat{\varepsilon}_{is}-\varepsilon_{is})\right| \\
&\lesssim \max_{r}\|F_r\|\max_{r}\frac{1}{T}\sum_{t=1}^{T}k_{h,rt}^{*}\|F_tF_t'\|\max_{i,t}\left|\frac{1}{T}\sum_{s=1}^{T}k_{h,st}^{*}F_s\left(\hat{\varepsilon}_{is}-\varepsilon_{is}\right)\right| \\
&= O_P(T^{1/(8+4\eta)})O_P\left(1\right)O_P(C_{NT}^{-2}\ln T), \\
\max_{i,r}\left|\frac{1}{T}\sum_{s=1}^{T}\xi_{h,sr}\varepsilon_{is}\right| &= \max_{i,r}\left|\frac{1}{T}\sum_{t=1}^{T}k_{h,rt}^{*}F_r'H_0H_0'F_tF_t'H_0H_0'\frac{1}{T}\sum_{s=1}^{T}k_{h,st}^{*}F_s\varepsilon_{is}\right| \\
&\lesssim \max_{r}\|F_r\|\max_{r}\frac{1}{T}\sum_{t=1}^{T}k_{h,rt}^{*}\|F_tF_t'\|\max_{i,t}\left|\frac{1}{T}\sum_{s=1}^{T}k_{h,st}^{*}F_s\varepsilon_{is}\right| \\
&= O_P(T^{1/(8+4\eta)})O_P\left(1\right)O_P((Th)^{-1/2}(\ln T)^{1/2}).
\end{aligned}$$

It follows that

$$\begin{aligned}
V_{12} &= \frac{h}{N}\sum_{i=1}^{N}\left[\frac{1}{T}\sum_{s,r=1}^{T}(\hat{\xi}_{h,sr}-\xi_{h,sr})\varepsilon_{is}(\hat{\varepsilon}_{ir}-\varepsilon_{ir})\right]^2 \\
&\le \frac{T^2h}{N}\sum_{i=1}^{N}\left[\max_{r}\left|\frac{1}{T}\sum_{s=1}^{T}(\hat{\xi}_{h,sr}-\xi_{h,sr})\varepsilon_{is}\right|\frac{1}{T}\sum_{r=1}^{T}|\hat{\varepsilon}_{ir}-\varepsilon_{ir}|\right]^2 \\
&\le T^2h\max_{i,r}\left|\frac{1}{T}\sum_{s=1}^{T}(\hat{\xi}_{h,sr}-\xi_{h,sr})\varepsilon_{is}\right|^2\frac{1}{N}\sum_{i=1}^{N}\left[\frac{1}{T}\sum_{r=1}^{T}|\hat{\varepsilon}_{ir}-\varepsilon_{ir}|\right]^2 \\
&= T^2hO_P(T^{1/(4+2\eta)}C_{NT}^{-4}(\ln T)^2)O_P\left(C_{NT}^{-2}\ln T\right) = o_P\left(1\right),
\end{aligned}$$

$$\begin{aligned}
V_{15} &= \frac{h}{N}\sum_{i=1}^{N}\left[\frac{1}{T}\sum_{s,r=1}^{T}\xi_{h,sr}(\hat{\varepsilon}_{ir}-\varepsilon_{ir})(\hat{\varepsilon}_{is}-\varepsilon_{is})\right]^2 \\
&\le T^2h\max_{i,r}\left|\frac{1}{T}\sum_{s=1}^{T}\xi_{h,sr}(\hat{\varepsilon}_{is}-\varepsilon_{is})\right|^2\frac{1}{N}\sum_{i=1}^{N}\left[\frac{1}{T}\sum_{r=1}^{T}|\hat{\varepsilon}_{ir}-\varepsilon_{ir}|\right]^2 \\
&= T^2hO_P(T^{1/(4+2\eta)}C_{NT}^{-4}(\ln T)^2)O_P\left(C_{NT}^{-2}\ln T\right) = o_P\left(1\right),
\end{aligned}$$

and

$$\begin{aligned}
V_{16} &= \frac{h}{N}\sum_{i=1}^{N}\left[\frac{1}{T}\sum_{s,r=1}^{T}\xi_{h,sr}(\hat{\varepsilon}_{ir}-\varepsilon_{ir})\varepsilon_{is}\right]^2 = T^2h\max_{i,r}\left|\frac{1}{T}\sum_{s=1}^{T}\xi_{h,sr}\varepsilon_{is}\right|^2\frac{1}{N}\sum_{i=1}^{N}\left[\frac{1}{T}\sum_{r=1}^{T}|\hat{\varepsilon}_{ir}-\varepsilon_{ir}|\right]^2 \\
&= T^2hO_P(T^{1/(4+2\eta)}C_{NT}^{-4}(\ln T)^2)O_P\left(C_{NT}^{-2}\ln T\right) = o_P\left(1\right).
\end{aligned}$$

For V_{14}, we use the decomposition in (B.5) to obtain

$$V_{14} = \frac{T^2h}{N}\sum_{i=1}^{N}\left[\frac{1}{T^2}\sum_{s,r=1}^{T}(\hat{\xi}_{h,sr}-\xi_{h,sr})\varepsilon_{is}\varepsilon_{ir}\right]^2 \le 6\sum_{l=1}^{6}\frac{T^2h}{N}\sum_{i=1}^{N}\left[\frac{1}{T^2}\sum_{s,r=1}^{T}\xi_{lh,sr}\varepsilon_{is}\varepsilon_{ir}\right]^2 \equiv 6\sum_{l=1}^{6}V_{14,l}.$$

We can prove $V_{14} = o_P(1)$ by showing that $V_{14,l} = o_P(1)$ for $l = 1, 2, ..., 6$. For example,

$$\begin{aligned}
V_{14,6} &= \frac{T^2 h}{N} \sum_{i=1}^{N} \left[\frac{1}{T^2} \sum_{s,r=1}^{T} \xi_{6h,sr} \varepsilon_{is} \varepsilon_{ir} \right]^2 \\
&= \frac{T^2 h}{N} \sum_{i=1}^{N} \left[\operatorname{tr} \left(H_0 H_0' \frac{1}{T} \sum_{t=1}^{T} F_t F_t' H_0 \frac{1}{T} \sum_{s=1}^{T} k_{h,st}^* (\tilde{F}_s - H_0' F_s) \varepsilon_{is} \frac{1}{T} \sum_{r=1}^{T} k_{h,rt}^* F_r' \varepsilon_{ir} \right) \right]^2 \\
&\lesssim T^2 h \max_{i,t} \left\| \frac{1}{T} \sum_{s=1}^{T} k_{h,st}^* (\tilde{F}_s - H_0' F_s) \varepsilon_{is} \right\|^2 \max_t \frac{1}{N} \sum_{i=1}^{N} \left\| \frac{1}{T} \sum_{r=1}^{T} k_{h,rt}^* \varepsilon_{ir} \right\|^2 \\
&= T^2 h O_P\left(C_{NT}^{-4} (\ln T)^2 \right) O_P\left((Th)^{-1} \ln T \right) = o_P(1).
\end{aligned}$$

In sum, we have shown that $V_1 = o_P(1)$.

Similarly, we can show that $V_2 = o_P(1)$. Next, noting that

$$V_3 = \bar{\mathbb{V}}_{NT}^{(1)} - \mathbb{V}_{NT}^{(1)} = \frac{h}{N} \sum_{i=1}^{N} \left(\frac{1}{T} \sum_{s,r=1}^{T} \xi_{h,sr} \varepsilon_{is} \varepsilon_{ir} \right)^2 - \frac{h}{N} \sum_{i=1}^{N} E_{\mathcal{C}} \left(\frac{1}{T} \sum_{s,r=1}^{T} \xi_{h,sr} \varepsilon_{is} \varepsilon_{ir} \right)^2,$$

we have $E_{\mathcal{C}}(V_3) = 0$ and

$$\begin{aligned}
\operatorname{Var}_{\mathcal{C}}(V_3) &= \frac{h^2}{N^2} \sum_{i=1}^{N} \operatorname{Var}_{\mathcal{C}} \left(\frac{1}{T} \sum_{s,r=1}^{T} \xi_{h,sr} \varepsilon_{is} \varepsilon_{ir} \right)^2 \le \frac{h^2}{N^2} \sum_{i=1}^{N} E_{\mathcal{C}} \left(\frac{1}{T} \sum_{s,r=1}^{T} \xi_{h,sr} \varepsilon_{is} \varepsilon_{ir} \right)^4 \\
&= \frac{h^2}{N^2} \sum_{i=1}^{N} E_{\mathcal{C}} \left(\frac{1}{T} \sum_{s,r=1}^{T} \xi_{h,sr} \varepsilon_{is} \varepsilon_{ir} \right)^4 = o_P(1).
\end{aligned}$$

Then $V_3 = o_P(1)$ by conditional Chebyshev inequality. So we have shown that $\hat{\mathbb{V}}_{NT}^{(1)} - \mathbb{V}_{NT}^{(1)} = o_P(1)$.

(ii) By Theorem 4.1(ii), it suffices to show (ii1) $\hat{\mathbb{B}}_{NT}^{(2)} - \mathbb{B}_{NT}^{(2)} = o_p(1)$ and (ii2) $\hat{\mathbb{V}}_{NT}^{(2)} - \mathbb{V}_{NT}^{(2)} = o_p(1)$. For the use in the proof of Theorem 4.3(ii), we prove the claims under $\mathbb{H}_1^{(2)}(a_{2NT})$.

We first prove (ii1) by showing that $\hat{\mathbb{B}}_{1,NT}^{(2)} - \mathbb{B}_{1,NT}^{(2)} = o_p(1)$ and $\hat{\mathbb{B}}_{2,NT}^{(2)} - \mathbb{B}_{2,NT}^{(2)} = o_p(1)$. We make the following two decompositions:

$$\begin{aligned}
& \hat{\varsigma}_{ij,sr} \hat{\varepsilon}_{jr} \hat{\varepsilon}_{is} - \xi_{h,sr} \varepsilon_{is} \varepsilon_{ir} \\
={}& (\hat{\varsigma}_{ij,sr} - \varsigma_{ij,sr})(\hat{\varepsilon}_{jr} \hat{\varepsilon}_{is} - \varepsilon_{is} \varepsilon_{jr}) + (\hat{\varsigma}_{ij,sr} - \varsigma_{ij,sr}) \varepsilon_{is} \varepsilon_{jr} + \varsigma_{ij,sr} (\hat{\varepsilon}_{jr} \hat{\varepsilon}_{is} - \varepsilon_{is} \varepsilon_{jr}) \\
={}& (\hat{\varsigma}_{ij,sr} - \varsigma_{ij,sr})(\hat{\varepsilon}_{jr} - \varepsilon_{jr})(\hat{\varepsilon}_{is} - \varepsilon_{is}) + (\hat{\varsigma}_{ij,sr} - \varsigma_{ij,sr})(\hat{\varepsilon}_{jr} - \varepsilon_{jr}) \varepsilon_{is} \\
& + (\hat{\varsigma}_{ij,sr} - \varsigma_{ij,sr})(\hat{\varepsilon}_{is} - \varepsilon_{is}) \varepsilon_{jr} + (\hat{\varsigma}_{ij,sr} - \varsigma_{ij,sr}) \varepsilon_{is} \varepsilon_{jr} \\
& + \varsigma_{ij,sr} (\hat{\varepsilon}_{jr} - \varepsilon_{jr})(\hat{\varepsilon}_{is} - \varepsilon_{is}) + \varsigma_{ij,sr} (\hat{\varepsilon}_{jr} - \varepsilon_{jr}) \varepsilon_{is} + \varsigma_{ij,sr} (\hat{\varepsilon}_{is} - \varepsilon_{is}) \varepsilon_{jr} \equiv \sum_{l=1}^{7} \phi_{l,ij,sr} \qquad \text{(B.6)}
\end{aligned}$$

and

$$\begin{aligned}
\hat{\varsigma}_{ii,sr} - \varsigma_{ii,sr} &= \frac{1}{T} \sum_{t=1}^{T} k_{h,st}^* k_{h,rt}^* \left[\hat{\mathcal{X}}_{is}^{(t)\prime} \hat{D}^{(t)} (\hat{F}^{(t)})^{-1} \hat{D}^{(t)} (\hat{F}^{(t)})^{-1} \hat{\mathcal{X}}_{ir}^{(t)} - \mathcal{X}_{is}^{(t)\prime} D (F^{(t)})^{-1} D (F^{(t)})^{-1} \mathcal{X}_{ir}^{(t)} \right] \\
&= \frac{1}{T} \sum_{t=1}^{T} k_{h,st}^* k_{h,rt}^* \{ [\hat{\mathcal{X}}_{is}^{(t)} - \mathcal{X}_{is}^{(t)}] [\hat{D}^{(t)} (\hat{F}^{(t)})^{-1} \hat{D}^{(t)} (\hat{F}^{(t)})^{-1} - D (F^{(t)})^{-1} D (F^{(t)})^{-1}] [\hat{\mathcal{X}}_{ir}^{(t)} - \mathcal{X}_{ir}^{(t)}]
\end{aligned}$$

$$+[\hat{\mathcal{X}}_{is}^{(t)} - \mathcal{X}_{is}^{(t)}][\hat{D}^{(t)}(\hat{F}^{(t)})^{-1}\hat{D}^{(t)}(\hat{F}^{(t)})^{-1} - D(F^{(t)})^{-1}D(F^{(t)})^{-1}]\mathcal{X}_{ir}^{(t)}$$
$$+[\hat{\mathcal{X}}_{is}^{(t)} - \mathcal{X}_{is}^{(t)}]D(F^{(t)})^{-1}D(F^{(t)})^{-1}[\hat{\mathcal{X}}_{ir}^{(t)} - \mathcal{X}_{ir}^{(t)}] + [\hat{\mathcal{X}}_{is}^{(t)} - \mathcal{X}_{is}^{(t)}]D(F^{(t)})^{-1}D(F^{(t)})^{-1}\mathcal{X}_{ir}^{(t)}$$
$$+\mathcal{X}_{is}^{(t)}[\hat{D}^{(t)}(\hat{F}^{(t)})^{-1}\hat{D}^{(t)}(\hat{F}^{(t)})^{-1} - D(F^{(t)})^{-1}D(F^{(t)})^{-1}][\hat{\mathcal{X}}_{ir}^{(t)} - \mathcal{X}_{ir}^{(t)}]$$
$$+\mathcal{X}_{is}^{(t)}[\hat{D}^{(t)}(\hat{F}^{(t)})^{-1}\hat{D}^{(t)}(\hat{F}^{(t)})^{-1} - D(F^{(t)})^{-1}D(F^{(t)})^{-1}]\mathcal{X}_{ir}^{(t)}$$
$$+\mathcal{X}_{is}^{(t)\prime}D(F^{(t)})^{-1}D(F^{(t)})^{-1}[\hat{\mathcal{X}}_{ir}^{(t)} - \mathcal{X}_{ir}^{(t)}]\} \equiv \sum_{l=1}^{7}\varsigma_{l,ii,sr}$$

and

$$\begin{aligned}
\hat{\varsigma}_{ii,sr} - \varsigma_{ii,sr} &= \frac{1}{T}\sum_{t=1}^{T}k_{h,st}^{*}k_{h,rt}^{*}\left[\hat{\mathcal{X}}_{is}^{(t)\prime}\hat{D}^{(t)}(\hat{F}^{(t)})^{-1}\hat{D}^{(t)}(\hat{F}^{(t)})^{-1}\hat{\mathcal{X}}_{ir}^{(t)} - \mathcal{X}_{is}^{(t)\prime}D(F^{(t)})^{-1}D(F^{(t)})^{-1}\mathcal{X}_{ir}^{(t)}\right] \\
&= \frac{1}{T}\sum_{t=1}^{T}k_{h,st}^{*}k_{h,rt}^{*}\{[\hat{\mathcal{X}}_{is}^{(t)} - \mathcal{X}_{is}^{(t)}][\hat{D}^{(t)}(\hat{F}^{(t)})^{-1}\hat{D}^{(t)}(\hat{F}^{(t)})^{-1} - D(F^{(t)})^{-1}D(F^{(t)})^{-1}][\hat{\mathcal{X}}_{ir}^{(t)} - \mathcal{X}_{ir}^{(t)}] \\
&\quad +[\hat{\mathcal{X}}_{is}^{(t)} - \mathcal{X}_{is}^{(t)}][\hat{D}^{(t)}(\hat{F}^{(t)})^{-1}\hat{D}^{(t)}(\hat{F}^{(t)})^{-1} - D(F^{(t)})^{-1}D(F^{(t)})^{-1}]\mathcal{X}_{ir}^{(t)} \\
&\quad +[\hat{\mathcal{X}}_{is}^{(t)} - \mathcal{X}_{is}^{(t)}]D(F^{(t)})^{-1}D(F^{(t)})^{-1}[\hat{\mathcal{X}}_{ir}^{(t)} - \mathcal{X}_{ir}^{(t)}] + [\hat{\mathcal{X}}_{is}^{(t)} - \mathcal{X}_{is}^{(t)}]D(F^{(t)})^{-1}D(F^{(t)})^{-1}\mathcal{X}_{ir}^{(t)} \\
&\quad +\mathcal{X}_{is}^{(t)}[\hat{D}^{(t)}(\hat{F}^{(t)})^{-1}\hat{D}^{(t)}(\hat{F}^{(t)})^{-1} - D(F^{(t)})^{-1}D(F^{(t)})^{-1}][\hat{\mathcal{X}}_{ir}^{(t)} - \mathcal{X}_{ir}^{(t)}] \\
&\quad +\mathcal{X}_{is}^{(t)}[\hat{D}^{(t)}(\hat{F}^{(t)})^{-1}\hat{D}^{(t)}(\hat{F}^{(t)})^{-1} - D(F^{(t)})^{-1}D(F^{(t)})^{-1}]\mathcal{X}_{ir}^{(t)} \\
&\quad +\mathcal{X}_{is}^{(t)\prime}D(F^{(t)})^{-1}D(F^{(t)})^{-1}[\hat{\mathcal{X}}_{ir}^{(t)} - \mathcal{X}_{ir}^{(t)}]\} \equiv \sum_{l=1}^{7}\varsigma_{l,ii,sr} \qquad \text{(B.7)}
\end{aligned}$$

Note that

$$\hat{\mathbb{B}}_{1,NT}^{(2)} - \mathbb{B}_{NT}^{(2)} = \frac{h^{1/2}}{NT}\sum_{i=1}^{N}\sum_{s,r=1}^{T}(\hat{\varepsilon}_{is}\hat{\varepsilon}_{ir}\hat{\varsigma}_{ii,sr} - \varepsilon_{is}\varepsilon_{ir}\varsigma_{ii,sr}) = \sum_{l=1}^{7}\frac{h^{1/2}}{NT}\sum_{i=1}^{N}\sum_{s,r=1}^{T}\phi_{l,ii,sr} \equiv \sum_{l=1}^{7}B_{l}^{(2)}$$

Noting that $B_2^{(2)} = B_3^{(2)}$ and $B_6^{(2)} = B_7^{(2)}$, it suffices to show $B_l^{(2)} = o_P(1)$ for $l = 1,\ 2,\ 4,\ 5,\ 6$. The proofs of these claims are straightforward so that we only demonstrate that $B_4^{(2)} = o_P(1)$ and $B_6^{(2)} = o_P(1)$. For $B_4^{(2)}$, we make the following decomposition:

$$B_4^{(2)} = \frac{h^{1/2}}{NT}\sum_{i=1}^{N}\sum_{s,r=1}^{T}(\hat{\varsigma}_{ii,sr} - \varsigma_{ii,sr})\varepsilon_{is}\varepsilon_{ir} = \sum_{l=1}^{7}\frac{h^{1/2}}{NT}\sum_{i=1}^{N}\sum_{s,r=1}^{T}\varsigma_{l,ii,sr}\varepsilon_{is}\varepsilon_{ir} \equiv \sum_{l=1}^{7}B_{4,l}^{(2)}.$$

We can show that $B_{4,l}^{(2)} = o_P(1)$ for $l = 1, ..., 7$. For example,

$$\begin{aligned}
\left|B_{4,7}^{(2)}\right| &= \frac{Th^{1/2}}{N}\left|\sum_{i=1}^{N}\frac{1}{T}\sum_{t=1}^{T}\operatorname{tr}\left\{D(F^{(t)})^{-1}D(F^{(t)})^{-1}\frac{1}{T}\sum_{r=1}^{T}k_{h,rt}^{*}[\hat{\mathcal{X}}_{ir}^{(t)} - \mathcal{X}_{ir}^{(t)}]\varepsilon_{ir}\frac{1}{T}\sum_{s=1}^{T}k_{h,st}^{*}\varepsilon_{is}\mathcal{X}_{is}^{(t)\prime}\right\}\right| \\
&\lesssim \frac{Th^{1/2}}{N}\sum_{i=1}^{N}\frac{1}{T}\sum_{t=1}^{T}\left\|\frac{1}{T}\sum_{r=1}^{T}k_{h,rt}^{*}[\hat{\mathcal{X}}_{ir}^{(t)} - \mathcal{X}_{ir}^{(t)}]\varepsilon_{ir}\right\|\left\|\frac{1}{T}\sum_{s=1}^{T}k_{h,st}^{*}\varepsilon_{is}\mathcal{X}_{is}^{(t)\prime}\right\| \\
&= Th^{1/2}O_P\left(C_{NT}^{-2}\ln T\right)O_P((Th)^{-1/2}) = o_P(1),
\end{aligned}$$

where we use the fact that $\max_{i,t}\left\|\frac{1}{T}\sum_{r=1}^{T}k_{h,rt}^{*}[\hat{\mathcal{X}}_{ir}^{(t)}-\mathcal{X}_{ir}^{(t)}]\varepsilon_{ir}\right\|=O_P\left(C_{NT}^{-2}\ln T\right)$ by straightforward verifications. For $B_6^{(2)}$, we have

$$\begin{aligned}\left|B_6^{(2)}\right| &= \frac{h^{1/2}}{NT}\left|\sum_{i=1}^{N}\sum_{s,r=1}^{T}\varsigma_{ii,sr}(\hat{\varepsilon}_{ir}-\varepsilon_{ir})\varepsilon_{is}\right| \\ &= \frac{Th^{1/2}}{N}\left|\sum_{i=1}^{N}\frac{1}{T}\sum_{t=1}^{T}\operatorname{tr}\left(D(F^{(t)})^{-1}D(F^{(t)})^{-1}\frac{1}{T}\sum_{r=1}^{T}k_{h,rt}^{*}\mathcal{X}_{ir}^{(t)}(\hat{\varepsilon}_{ir}-\varepsilon_{ir})\frac{1}{T}\sum_{s=1}^{T}k_{h,st}^{*}\varepsilon_{is}\mathcal{X}_{is}^{(t)\prime}\right)\right| \\ &\lesssim \frac{Th^{1/2}}{N}\sum_{i=1}^{N}\frac{1}{T}\sum_{t=1}^{T}\left\|\frac{1}{T}\sum_{r=1}^{T}k_{h,rt}^{*}\mathcal{X}_{ir}^{(t)}(\hat{\varepsilon}_{ir}-\varepsilon_{ir})\right\|\left\|\frac{1}{T}\sum_{s=1}^{T}k_{h,st}^{*}\varepsilon_{is}\mathcal{X}_{is}^{(t)\prime}\right\| \\ &= Th^{1/2}O_P\left(C_{NT}^{-2}\ln T\right)O_P((Th)^{-1/2})=o_P(1).\end{aligned}$$

Consequently, we have established that $\hat{\mathbb{B}}_{1,NT}^{(2)}-\mathbb{B}_{1,NT}^{(2)}=o_p(1)$.

Next,

$$\begin{aligned}\hat{\mathbb{B}}_{2,NT}^{(2)}-\mathbb{B}_{2,NT}^{(2)} &= \frac{2h^{1/2}}{NT^2}\sum_{1\le j<i\le N}\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,st}^{*2}[\hat{\varepsilon}_{is}\hat{\varepsilon}_{js}\hat{\mathcal{X}}_{is}^{(t)\prime}\hat{\mathbb{D}}^{(t)}\hat{\mathcal{X}}_{js}^{(t)}-\varepsilon_{is}\varepsilon_{js}\mathcal{X}_{is}^{(t)\prime}\mathbb{D}^{(t)}\mathcal{X}_{js}^{(t)}] \\ &= \frac{2h^{1/2}}{NT^2}\sum_{1\le j<i\le N}\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,st}^{*2}\{[\hat{\varepsilon}_{is}\hat{\varepsilon}_{js}-\varepsilon_{is}\varepsilon_{js}]\mathcal{X}_{is}^{(t)\prime}\mathbb{D}^{(t)}\mathcal{X}_{js}^{(t)}+\varepsilon_{is}\varepsilon_{js}[\hat{\mathcal{X}}_{is}^{(t)\prime}\hat{\mathbb{D}}^{(t)}\hat{\mathcal{X}}_{js}^{(t)}-\mathcal{X}_{is}^{(t)\prime}\mathbb{D}^{(t)}\mathcal{X}_{js}^{(t)}] \\ &\quad +[\hat{\varepsilon}_{is}\hat{\varepsilon}_{js}-\varepsilon_{is}\varepsilon_{js}][\hat{\mathcal{X}}_{is}^{(t)\prime}\hat{\mathbb{D}}^{(t)}\hat{\mathcal{X}}_{js}^{(t)}-\mathcal{X}_{is}^{(t)\prime}\mathbb{D}^{(t)}\mathcal{X}_{js}^{(t)}]\}\equiv 2B_{2,1}+2B_{2,2}+B_{2,3}.\end{aligned}$$

For $B_{2,1}$, we make the following decomposition.

$$\begin{aligned}B_{2,1} &= \frac{2h^{1/2}}{NT^2}\sum_{1\le j<i\le N}\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,st}^{*2}\{[\hat{\varepsilon}_{is}\hat{\varepsilon}_{js}-\varepsilon_{is}\varepsilon_{js}]\mathcal{X}_{is}^{(t)\prime}\mathbb{D}^{(t)}\mathcal{X}_{js}^{(t)}+\varepsilon_{is}\varepsilon_{js}[\hat{\mathcal{X}}_{is}^{(t)\prime}\hat{\mathbb{D}}^{(t)}\hat{\mathcal{X}}_{js}^{(t)}-\mathcal{X}_{is}^{(t)\prime}\mathbb{D}^{(t)}\mathcal{X}_{js}^{(t)}] \\ &\equiv B_{2,11}+B_{2,12}.\end{aligned}$$

Noting that $\hat{\varepsilon}_{is}\hat{\varepsilon}_{js}-\varepsilon_{is}\varepsilon_{js}=(\hat{\varepsilon}_{is}-\varepsilon_{is})(\hat{\varepsilon}_{js}-\varepsilon_{js})+(\hat{\varepsilon}_{is}-\varepsilon_{is})\varepsilon_{js}+\varepsilon_{is}(\hat{\varepsilon}_{js}-\varepsilon_{js})$, we have

$$\begin{aligned}B_{2,11} &= \frac{2h^{1/2}}{NT^2}\sum_{1\le j<i\le N}\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,st}^{*2}[\hat{\varepsilon}_{is}\hat{\varepsilon}_{js}-\varepsilon_{is}\varepsilon_{js}]\mathcal{X}_{is}^{(t)\prime}\mathbb{D}^{(t)}\mathcal{X}_{js}^{(t)} \\ &= \frac{2h^{1/2}}{NT^2}\sum_{1\le j<i\le N}\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,st}^{*2}[(\hat{\varepsilon}_{is}-\varepsilon_{is})(\hat{\varepsilon}_{js}-\varepsilon_{js})+(\hat{\varepsilon}_{is}-\varepsilon_{is})\varepsilon_{js}+\varepsilon_{is}(\hat{\varepsilon}_{js}-\varepsilon_{js})]\mathcal{X}_{is}^{(t)\prime}\mathbb{D}^{(t)}\mathcal{X}_{js}^{(t)}\equiv\sum_{l=1}^{3}B_{2,11l}.\end{aligned}$$

We can readily show that $B_{2,11}=o_P(1)$ by showing that $B_{2,11l}=o_P(1)$ for $l=1,2,3$. For example,

$$\begin{aligned}|B_{2,112}| &= \left|\frac{h^{1/2}}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,st}^{*2}(\hat{\varepsilon}_{is}-\varepsilon_{is})\mathcal{X}_{is}^{(t)\prime}\mathbb{D}^{(t)}\sum_{j=1}^{N}\mathcal{X}_{js}^{(t)}\varepsilon_{js}\right|+o_P(1) \\ &\le \left|\frac{h^{1/2}}{NT^2}\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,st}^{*2}\sum_{i=1}^{N}(\hat{\varepsilon}_{is}-\varepsilon_{is})\mathcal{X}_{is}^{(t)\prime}\mathbb{D}^{(t)}\sum_{j=1}^{N}\mathcal{X}_{js}^{(t)}\varepsilon_{js}\right|+o_P(1) \\ &\lesssim N\left\{\frac{h}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,st}^{*2}\left\|\frac{1}{N}\sum_{i=1}^{N}(\hat{\varepsilon}_{is}-\varepsilon_{is})\mathcal{X}_{is}^{(t)\prime}\right\|^2\right\}^{1/2}\left\{\frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,st}^{*2}\left\|\frac{1}{N}\sum_{j=1}^{N}\mathcal{X}_{js}^{(t)}\varepsilon_{js}\right\|^2\right\}^{1/2}+o_P(1)\end{aligned}$$

$$= \quad NO_P((NTh)^{-1/2})O_P(h^{-1/2}N^{-1/2}) + o_P(1) = O_P(T^{-1/2}h^{-1}) + o_P(1) = o_P(1),$$

as we can show that $\frac{h}{T^2}\sum_{t=1}^T\sum_{s=1}^T k_{h,st}^{*2}\left\|\frac{1}{N}\sum_{i=1}^N(\hat{\varepsilon}_{is}-\varepsilon_{is})\mathcal{X}_{is}^{(t)\prime}\right\|^2 = O_P\left((NTh)^{-1}+h^4\right) = O_P\left((NTh)^{-1}\right).$ Similarly, we can show that $B_{2,12} = o_P(1)$.

For $B_{2,2}$, we make the following decomposition:

$$\begin{aligned}
B_{2,2} &= \frac{2h^{1/2}}{NT^2}\sum_{1\le j<i\le N}\sum_{t=1}^T\sum_{s=1}^T k_{h,st}^{*2}\varepsilon_{is}\varepsilon_{js}[\hat{\mathcal{X}}_{is}^{(t)\prime}\hat{\mathbb{D}}^{(t)}\hat{\mathcal{X}}_{js}^{(t)} - \mathcal{X}_{is}^{(t)\prime}\mathbb{D}^{(t)}\mathcal{X}_{js}^{(t)}] \\
&= \frac{2h^{1/2}}{NT^2}\sum_{1\le j<i\le N}\sum_{t=1}^T\sum_{s=1}^T k_{h,st}^{*2}\varepsilon_{is}\varepsilon_{js}\{[\hat{\mathcal{X}}_{is}^{(t)}-\mathcal{X}_{is}^{(t)}]'[\hat{\mathbb{D}}^{(t)}-\mathbb{D}^{(t)}][\hat{\mathcal{X}}_{js}^{(t)}-\mathcal{X}_{js}^{(t)}] \\
&\quad +[\hat{\mathcal{X}}_{is}^{(t)}-\mathcal{X}_{is}^{(t)}]'[\hat{\mathbb{D}}^{(t)}-\mathbb{D}^{(t)}]\mathcal{X}_{js}^{(t)} + [\hat{\mathcal{X}}_{is}^{(t)}-\mathcal{X}_{is}^{(t)}]'\mathbb{D}^{(t)}[\hat{\mathcal{X}}_{js}^{(t)}-\mathcal{X}_{js}^{(t)}] \\
&\quad +[\hat{\mathcal{X}}_{is}^{(t)}-\mathcal{X}_{is}^{(t)}]'\mathbb{D}^{(t)}\mathcal{X}_{js}^{(t)} + \mathcal{X}_{is}^{(t)\prime}(\hat{\mathbb{D}}^{(t)}-\mathbb{D}^{(t)})(\hat{\mathcal{X}}_{js}^{(t)}-\mathcal{X}_{js}^{(t)}) + \mathcal{X}_{is}^{(t)\prime}(\hat{\mathbb{D}}^{(t)}-\mathbb{D}^{(t)})\mathcal{X}_{js}^{(t)} \\
&\quad +\mathcal{X}_{is}^{(t)\prime}\mathbb{D}^{(t)}[\hat{\mathcal{X}}_{js}^{(t)}-\mathcal{X}_{js}^{(t)}]\} \equiv \sum_{l=1}^7 B_{2,2l}.
\end{aligned}$$

It is easy to show that $B_{2,2l} = o_P(1)$ for $l = 1$, 2, 3, 5, and 6. Next, $B_{2,24} = \bar{B}_{2,24} + o_P(1)$, where $\bar{B}_{2,24} = \frac{h^{1/2}}{NT^2}\sum_{i,j}\sum_{t=1}^T\sum_{s=1}^T k_{h,st}^{*2}\varepsilon_{is}\varepsilon_{js}[\hat{\mathcal{X}}_{is}^{(t)}-\mathcal{X}_{is}^{(t)}]'\mathbb{D}^{(t)}\mathcal{X}_{js}^{(t)}$. Note that

$$\begin{aligned}
|\bar{B}_{2,24}| &= \left|\frac{Nh^{1/2}}{T^2}\sum_{t=1}^T\sum_{s=1}^T k_{h,st}^{*2}\left\{\frac{1}{N}\sum_{i=1}^N\varepsilon_{is}[\hat{\mathcal{X}}_{is}^{(t)}-\mathcal{X}_{is}^{(t)}]'\right\}\mathbb{D}^{(t)}\left\{\frac{1}{N}\sum_{j=1}^N\mathcal{X}_{js}^{(t)}\varepsilon_{js}\right\}\right| \\
&\lesssim N\left\{\frac{h}{T^2}\sum_{t=1}^T\sum_{s=1}^T k_{h,st}^{*2}\left\|\frac{1}{N}\sum_{i=1}^N\varepsilon_{is}[\hat{\mathcal{X}}_{is}^{(t)}-\mathcal{X}_{is}^{(t)}]'\right\|^2\right\}^{1/2}\left\{\frac{1}{T^2}\sum_{t=1}^T\sum_{s=1}^T k_{h,st}^{*2}\left\|\frac{1}{N}\sum_{j=1}^N\mathcal{X}_{js}^{(t)}\varepsilon_{js}\right\|^2\right\}^{1/2} \\
&= NO_P((NTh)^{-1/2})O_P((Nh)^{-1/2}) = o_P(1)
\end{aligned}$$

Similarly, $B_{2,27} = o_P(1)$. Then $B_{2,2} = o_P(1)$. Lastly, one can also show that $B_{2,3} = o_P(1)$. Then $\hat{\mathbb{B}}_{2,NT}^{(2)} - \mathbb{B}_{2,NT}^{(2)} = o_P(1)$.

Let $\bar{\mathbb{V}}_{NT}^{(2)} = \frac{4h}{N^2T^2}\sum_{1\le i<j\le N}\left(\sum_{s,r=1}^T\varepsilon_{is}\varepsilon_{jr}\varsigma_{ij,sr}\right)^2$. Note that

$$\begin{aligned}
\hat{\mathbb{V}}_{NT}^{(2)} - \mathbb{V}_{NT}^{(2)} &= \frac{2h}{N^2}\sum_{1\le i\ne j\le N}\left[\left(\frac{1}{T}\sum_{s\ne r}\hat{\varepsilon}_{is}\hat{\varepsilon}_{jr}\hat{\varsigma}_{ij,sr}\right)^2 - \left(\frac{1}{T}\sum_{s\ne r}\varepsilon_{is}\varepsilon_{jr}\varsigma_{ij,sr}\right)^2\right] + [\bar{\mathbb{V}}_{NT}^{(2)} - \mathbb{V}_{NT}^{(2)}] \\
&= \frac{2h}{N^2}\sum_{1\le i\ne j\le N}\left(\frac{1}{T}\sum_{s\ne r}[\hat{\varepsilon}_{is}\hat{\varepsilon}_{jr}\hat{\varsigma}_{ij,sr} - \varepsilon_{is}\varepsilon_{jr}\varsigma_{ij,sr}]\right)^2 \\
&\quad + \frac{4h}{N^2}\sum_{1\le i\ne j\le N}\left(\frac{1}{T}\sum_{s\ne r}[\hat{\varepsilon}_{is}\hat{\varepsilon}_{jr}\hat{\varsigma}_{ij,sr} - \varepsilon_{is}\varepsilon_{jr}\varsigma_{ij,sr}]\right)\frac{1}{T}\sum_{s\ne r}\varepsilon_{is}\varepsilon_{jr}\varsigma_{ij,sr} + [\bar{\mathbb{V}}_{NT}^{(2)} - \mathbb{V}_{NT}^{(2)}] \\
&\equiv 2V_4 + 4V_5 + V_6.
\end{aligned}$$

Note that $V_4 \leq 7\sum_{l=1}^{7}\frac{h}{N^2}\sum_{1\leq i\neq j\leq N}\left(\frac{1}{T}\sum_{s\neq r}\phi_{l,ij,sr}\right)^2 \equiv 7\sum_{l=1}^{7}V_{4,l}$. We can readily show that $V_4 = o_P(1)$ by proving that $V_{4,l} = o_P(1)$ for $l = 1, ..., 7$. For example,

$$\begin{aligned}
V_{4,7} &= \frac{h}{N^2}\sum_{1\leq i\neq j\leq N}\left(\frac{1}{T}\sum_{s\neq r}\varsigma_{ij,sr}(\hat{\varepsilon}_{is}-\varepsilon_{is})\varepsilon_{jr}\right)^2 \\
&= \frac{h}{N^2}\sum_{1\leq i\neq j\leq N}\left(\frac{1}{T}\sum_{s\neq r}\frac{1}{T}\sum_{t=1}^{T}k^*_{h,st}k^*_{h,rt}\mathcal{X}_{is}^{(t)\prime}D(F^{(t)})^{-1}D(F^{(t)})^{-1}\mathcal{X}_{jr}^{(t)}(\hat{\varepsilon}_{is}-\varepsilon_{is})\varepsilon_{jr}\right)^2 \\
&= \frac{T^2h}{N^2}\sum_{1\leq i\neq j\leq N}\left(\frac{1}{T}\sum_{t=1}^{T}\frac{1}{T}\sum_{s=1}^{T}k^*_{h,st}(\hat{\varepsilon}_{is}-\varepsilon_{is})\mathcal{X}_{is}^{(t)\prime}D(F^{(t)})^{-1}D(F^{(t)})^{-1}\frac{1}{T}\sum_{s\neq r}k^*_{h,rt}\mathcal{X}_{jr}^{(t)}\varepsilon_{jr}\right)^2 \\
&\lesssim \frac{T^2h}{N^2}\sum_{1\leq i\neq j\leq N}\frac{1}{T}\sum_{t=1}^{T}\left\|\frac{1}{T}\sum_{s=1}^{T}k^*_{h,st}(\hat{\varepsilon}_{is}-\varepsilon_{is})\mathcal{X}_{is}^{(t)\prime}\right\|^2\frac{1}{T}\sum_{t=1}^{T}\left\|\frac{1}{T}\sum_{s\neq r}k^*_{h,rt}\mathcal{X}_{jr}^{(t)}\varepsilon_{jr}\right\|^2 \\
&= T^2hO_P(C_{NT}^{-4}(\ln T)^2)O_P((Th)^{-1}) = o_P(1).
\end{aligned}$$

For V_5, we apply CS inequality to obtain

$$\begin{aligned}
V_5 &= \frac{h}{N^2}\sum_{1\leq i\neq j\leq N}\left(\frac{1}{T}\sum_{s\neq r}[\hat{\varepsilon}_{is}\hat{\varepsilon}_{jr}\hat{\varsigma}_{ij,sr}-\varepsilon_{is}\varepsilon_{jr}\varsigma_{ij,sr}]\right)\frac{1}{T}\sum_{s\neq r}\varepsilon_{is}\varepsilon_{jr}\varsigma_{ij,sr} \\
&\leq \{V_4\}^{1/2}\left\{\frac{h}{N^2}\sum_{1\leq i\neq j\leq N}\left(\frac{1}{T}\sum_{s\neq r}\varepsilon_{is}\varepsilon_{jr}\varsigma_{ij,sr}\right)^2\right\}^{1/2} = o_P(1)\,O_P(1) = o_P(1),
\end{aligned}$$

where we also use the fact that

$$\begin{aligned}
&\frac{h}{N^2}\sum_{1\leq i\neq j\leq N}E_{\mathcal{C}}\left(\frac{1}{T}\sum_{s\neq r}\varepsilon_{is}\varepsilon_{jr}\varsigma_{ij,sr}\right)^2 \\
\leq\; &\frac{2h}{N^2}\sum_{1\leq i\neq j\leq N}\left(\frac{1}{T}\sum_{s\neq r}E_{\mathcal{C}}\left(\varepsilon_{is}\varepsilon_{jr}\varsigma_{ij,sr}\right)\right)^2+\frac{2h}{N^2}\sum_{1\leq i\neq j\leq N}\mathrm{Var}_{\mathcal{C}}\left(\frac{1}{T}\sum_{s\neq r}\varepsilon_{is}\varepsilon_{jr}\varsigma_{ij,sr}\right)^2 = O_P(1)+O_P(1)
\end{aligned}$$

and

$$\begin{aligned}
\left|\frac{1}{T}\sum_{s\neq r}E_{\mathcal{C}}\left(\varepsilon_{is}\varepsilon_{jr}\varsigma_{ij,sr}\right)\right| &= \left|\frac{1}{T}\sum_{s\neq r}\frac{1}{T}\sum_{t=1}^{T}k^*_{h,st}k^*_{h,rt}E_{\mathcal{C}}\left[\varepsilon_{is}\mathcal{X}_{is}^{(t)\prime}D(F^{(t)})^{-1}D(F^{(t)})^{-1}\mathcal{X}_{jr}^{(t)}\varepsilon_{jr}\right]\right| \\
&\lesssim \left|\frac{1}{T}\sum_{s\neq r}\frac{1}{T}\sum_{t=1}^{T}k^*_{h,st}k^*_{h,rt}\mathrm{tr}\left(E_{\mathcal{C}}\left[\mathcal{X}_{jr}^{(t)}\varepsilon_{jr}\varepsilon_{is}\mathcal{X}_{is}^{(t)\prime}\right]\right)\right| \\
&\asymp \left|\frac{1}{T}\sum_{s\neq r}\frac{1}{T}\sum_{t=1}^{T}k^*_{h,st}k^*_{h,rt}E_{\mathcal{C}}\left[\varepsilon_{is}\mathcal{X}_{is}'\mathcal{X}_{jr}\varepsilon_{jr}\right]\right| \\
&\leq \frac{1}{T}\sum_{r=1}^{T}\frac{1}{T}\sum_{t=1}^{T}\max_s k^*_{h,st}k^*_{h,rt}\sum_{s=1}^{T}\left|E_{\mathcal{C}}\left[\varepsilon_{is}\mathcal{X}_{is}'\mathcal{X}_{jr}\varepsilon_{jr}\right]\right|
\end{aligned}$$

$$\leq \max_{i,j,r} \sum_{s=1}^{T} |E_{\mathcal{C}}[\varepsilon_{is}\mathcal{X}_{is}'\mathcal{X}_{jr}\varepsilon_{jr}]| \frac{1}{T}\sum_{t=1}^{T}\max_s k_{h,st}^* \frac{1}{T}\sum_{r=1}^{T} k_{h,rt}^* = O_P(1).$$

Note that $V_6 = \frac{4h}{N^2T^2}\sum_{1\leq i<j\leq N}\left[\left(\sum_{s\neq r}\varepsilon_{is}\varepsilon_{jr}\varsigma_{ij,sr}\right)^2 - E_{\mathcal{C}}\left(\sum_{s\neq r}\varepsilon_{is}\varepsilon_{jr}\varsigma_{ij,sr}\right)^2\right]$. It is easy to see that $E_{\mathcal{C}}(V_6) = 0$ and

$$\mathrm{Var}_{\mathcal{C}}(V_6) = \frac{16h^2}{N^4}\sum_{1\leq i<j\leq N}\mathrm{Var}_{\mathcal{C}}\left(\left(\frac{1}{T}\sum_{s\neq r}\varepsilon_{is}\varepsilon_{jr}\varsigma_{ij,sr}\right)^2\right) \leq \frac{16h^2}{N^4}\sum_{1\leq i<j\leq N} E_{\mathcal{C}}\left[\left(\frac{1}{T}\sum_{s\neq r}\varepsilon_{is}\varepsilon_{jr}\varsigma_{ij,sr}\right)^4\right] = o_P(1).$$

(iii) The estimators are the same as those in (i) and the proof is similar to that of (i) and thus omitted. ■

Proof of Theorem 4.3. By the proof of Theorem 4.1, we have $J_{NT}^{(l)} \xrightarrow{d} N(\pi^{(l)}, 1)$ under the corresponding local alternatives for $l = 1, 2, 3$. By the proof of Theorem 4.2, we have $\hat{\mathbb{B}}_{NT}^{(l)} = \mathbb{B}_{NT}^{(l)} + o_P(1)$, and $\hat{\mathbb{V}}_{NT}^{(l)} = \mathbb{V}_{NT}^{(l)} + o_P(1)$ under the corresponding local alternatives for $l = 1, 2, 3$. It follows that $\hat{J}_{NT}^{(l)} \xrightarrow{d} N(\pi^{(l)}, 1)$ under the corresponding local alternatives. ■

Proof of Theorem 4.4. The proof of this theorem is almost the same as that of Theorem 4.5 in Su and Wang (2017). We do not repeat it here. ■

C Proofs of the Technical Lemmas in Appendix A

Proof of Lemma A.1. (i) Note that $\frac{1}{NT}\sum_{i=1}^{N} X_i^{(r)\prime} M_{F^{(r)}}\varepsilon_i^{(r)} = \frac{1}{NT}\sum_{i=1}^{N} X_i^{(r)\prime}\varepsilon_i^{(r)} - \frac{1}{NT}\sum_{i=1}^{N} X_i^{(r)\prime} P_{F^{(r)}}\varepsilon_i^{(r)}$. For the first term, we can apply Chebyshev inequality to show that $\frac{1}{NT}\sum_{i=1}^{N} X_i^{(r)\prime}\varepsilon_i^{(r)} = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} k_{tr}^* X_{it}\varepsilon_{it}$ $= O_P((NTh)^{-1/2})$. For the second term, noting that $P_{F^{(r)}} = T^{-1}F^{(r)}F^{(r)\prime}$, we have

$$\frac{1}{NT}\left\|\sum_{i=1}^{N} X_i^{(r)\prime} P_{F^{(r)}}\varepsilon_i^{(r)}\right\| = \left\|\frac{1}{N}\sum_{i=1}^{N}\frac{X_i^{(r)\prime}F^{(r)}}{T}\frac{1}{T}\sum_{t=1}^{T} F_t^{(r)}\varepsilon_{it}^{(r)}\right\| \leq \frac{1}{N}\sum_{i=1}^{N}\left\|\frac{X_i^{(r)\prime}F^{(r)}}{T}\right\|\left\|\frac{1}{T}\sum_{t=1}^{T}F_t^{(r)}\varepsilon_{it}^{(r)}\right\|$$
$$\leq \left(\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{X_i^{(r)\prime}F^{(r)}}{T}\right\|^2\right)^{1/2}\left(\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{t=1}^{T}F_t^{(r)}\varepsilon_{it}^{(r)}\right\|^2\right)^{1/2},$$

where the first inequality holds by the submultiplicative property of the Frobenius norm and the second inequality holds by Cauchy-Schwarz (CS) inequality. It is easy to apply Markov inequality to show that

$$\sup_{F^{(r)}\in\mathcal{F}}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{X_i^{(r)\prime}F^{(r)}}{T}\right\|^2 \leq \sup_{F^{(r)}\in\mathcal{F}}\frac{1}{T}\left\|F^{(r)}\right\|^2\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left\|X_{it}^{(r)}\right\|^2 = \frac{R}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}k_{tr}^*\left\|X_{it}\right\|^2 = O_P(1).$$

In addition

$$\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{t=1}^{T}F_t^{(r)}\varepsilon_{it}^{(r)}\right\|^2 = \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}F_t^{(r)\prime}F_t^{(r)}(\varepsilon_{it}^{(r)})^2 + \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t\neq s}F_t^{(r)\prime}F_s^{(r)}\varepsilon_{it}^{(r)}\varepsilon_{is}^{(r)} \equiv I_1 + I_2.$$

For I_1, we have $I_1 \leq \sup_{F^{(r)}\in\mathcal{F}}\frac{1}{T^2}\sum_{t=1}^{T}\left\|F_t^{(r)}\right\|^2\max_t\frac{1}{N}\sum_{i=1}^{N}k_{tr}^*\varepsilon_{it}^2 \leq \frac{R}{T}\max_t\frac{1}{N}\sum_{i=1}^{N}k_{tr}^*\varepsilon_{it}^2 = O_P\left(T^{-1}\right)$ as it is standard to show that $\max_t\frac{1}{N}\sum_{i=1}^{N}k_{tr}^*\varepsilon_{it}^2 \leq \max_t\frac{1}{N}\sum_{i=1}^{N}k_{tr}^*E(\varepsilon_{it}^2) + \max_t\frac{1}{N}\sum_{i=1}^{N}k_{tr}^*[\varepsilon_{it}^2 - E(\varepsilon_{it}^2)] = O_P(1)$.

For I_2, we have by CS inequality:

$$
\begin{aligned}
&\sup_{F^{(r)}\in\mathcal{F}}\left\|\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t\neq s}F_t^{(r)\prime}F_s^{(r)}\varepsilon_{it}^{(r)}\varepsilon_{is}^{(r)}\right\| \\
&=\sup_{F^{(r)}\in\mathcal{F}}\left\|\frac{1}{T^2}\sum_{t\neq s}F_t^{(r)\prime}F_s^{(r)}\left(k_{tr}^{*}k_{sr}^{*}\right)^{1/2}\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{it}\varepsilon_{is}\right\| \\
&\leq N^{-1/2}\left(\sup_{F^{(r)}\in\mathcal{F}}\frac{1}{T^2}\sum_{t\neq s}\left\|F_s^{(r)}\right\|^2\left\|F_s^{(r)}\right\|^2\right)^{1/2}\left(\frac{1}{T^2}\sum_{t\neq s}\left(k_{tr}^{*}k_{sr}^{*}\right)^{1/2}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\varepsilon_{it}\varepsilon_{is}\right]^2\right)^{1/2}=O_P(N^{-1/2}),
\end{aligned}
$$

where we use the fact that $\frac{1}{T^2}\sum_{t\neq s}\left(k_{tr}^{*}k_{sr}^{*}\right)^{1/2}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\varepsilon_{it}\varepsilon_{is}\right]^2=O_P\left(1\right)$ by Markov equality and Assumption A.2(i)-(iii). It follows that

$$
\sup_{F^{(r)}\in\mathcal{F}}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{t=1}^{T}F_t^{(r)}\varepsilon_{it}^{(r)}\right\|^2=O_P(T^{-1/2}+N^{-1/4}) \tag{C.1}
$$

and $\sup_{F^{(r)}\in\mathcal{F}}\frac{1}{NT}\left\|\sum_{i=1}^{N}X_i^{(r)\prime}M_{F^{(r)}}\varepsilon_i^{(r)}\right\|=O_P((NTh)^{-1/2}+T^{-1/2}+N^{-1/4})=O_P(T^{-1/2}+N^{-1/4})$.

(ii) and (iii) The proof is similar to that of (i) and thus omitted here.

(iv) Note that $\sup_{F^{(r)}\in\mathcal{F}}\frac{1}{NT}\left\|\sum_{i=1}^{N}\Delta_i^{(r)\prime}(P_{F^{(r)}}-P_{F^{(r)0}})\Delta_i^{(r)}\right\|\leq\frac{1}{NT}\sum_{i=1}^{N}\left\|\Delta_i^{(r)\prime}\Delta_i^{(r)}\right\|$. The result follows because

$$
\frac{1}{NT}\sum_{i=1}^{N}\left\|\Delta_i^{(r)\prime}\Delta_i^{(r)}\right\|=\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}k_{h,tr}^{*}\left[X_{it}^{\prime}d_0\left(t,r\right)+F_t^{0}d_i\left(t,r\right)\right]^2=O_P(h^2). \tag{C.2}
$$

(v) Note that

$$
\sup_{F^{(r)}\in\mathcal{F}}\left\|\frac{1}{NT}\sum_{i=1}^{N}X_i^{(r)\prime}M_{F^{(r)}}\Delta_i^{(r)}\right\|\leq\left\{\frac{1}{NT}\sum_{i=1}^{N}\left\|X_i^{(r)}\right\|^2\right\}^{1/2}\left\{\frac{1}{NT}\sum_{i=1}^{N}\left\|\Delta_i^{(r)}\right\|^2\right\}^{1/2}=O_P(1)O_P(h)=O_P(h).
$$

(vi) The proof is quite similar to that of (v) and thus omitted here.

(vii) We only show that $\sup_{F^{(r)}\in\mathcal{F}}\frac{1}{NT}\left\|\sum_{i=1}^{N}\Delta_i^{(r)\prime}P_{F^{(r)}}\varepsilon_i^{(r)}\right\|=O_P((T^{-1/2}+N^{-1/4})h)$ as it is easier to show that $\frac{1}{NT}\left\|\sum_{i=1}^{N}\Delta_i^{(r)\prime}P_{F^{(r)0}}\varepsilon_i^{(r)}\right\|=O_P((T^{-1/2}+N^{-1/4})h)$. By (C.1) and (C.2),

$$
\begin{aligned}
\sup_{F^{(r)}\in\mathcal{F}}\frac{1}{NT}\left\|\sum_{i=1}^{N}\Delta_i^{(r)\prime}P_{F^{(r)}}\varepsilon_i^{(r)}\right\| &= \sup_{F^{(r)}\in\mathcal{F}}\frac{1}{NT^2}\left\|\sum_{i=1}^{N}\Delta_i^{(r)\prime}F^{(r)}F^{(r)\prime}\varepsilon_i^{(r)}\right\| \\
&\leq \sup_{F^{(r)}\in\mathcal{F}}T^{-1/2}\left\|F^{(r)}\right\|\left\{\frac{1}{NT}\sum_{i=1}^{N}\left\|\Delta_i^{(r)}\right\|^2\right\}^{1/2}\sup_{F^{(r)}\in\mathcal{F}}\left\{\frac{1}{NT^2}\sum_{i=1}^{N}\left\|F^{(r)\prime}\varepsilon_i^{(r)}\right\|^2\right\}^{1/2} \\
&\leq R^{1/2}O_P(h)O_P(T^{-1/2}+N^{-1/4})=O_P((T^{-1/2}+N^{-1/4})h).
\end{aligned}
$$

It follows that $\sup_{F^{(r)}}\frac{1}{NT}\left\|\sum_{i=1}^{N}\Delta_i^{(r)\prime}(P_{F^{(r)}}-P_{F^{(r)0}})\varepsilon_i^{(r)}\right\|=O_P((T^{-1/2}+N^{-1/4})h)$. ■

In the following proofs, we suppress the superscript 0 for the true parameters β_r^0, $F_t^{(r)0}$, $F^{(r)0}$, λ_{ir}^0 and Λ_r^0 unless confusion may arise.

Proof of Lemma A.2. This Lemma is parallel to Lemma A.1 in Su and Wang (2017) and we try to be brief in the proof.

(i) Noting that $Y_i^{(r)} - X_i^{(r)}\hat{\beta}_r = -X_i^{(r)}\hat{\delta}_r + F^{(r)}\lambda_{ir} + \Delta_i^{(r)} + \varepsilon_i^{(r)}$ with $\hat{\delta}_r = \hat{\beta}_r - \beta_r$, we start with the eigenvalue problem $\left[\frac{1}{NT}(Y^{(r)} - X^{(r)}\hat{\beta}_r)(Y^{(r)} - X^{(r)}\hat{\beta}_r)'\right]\hat{F}^{(r)} = \hat{F}^{(r)}\hat{V}_{NT}^{(r)}$ to obtain the following decomposition:

$$
\begin{aligned}
&\hat{F}^{(r)}\hat{V}_{NT}^{(r)} - \frac{1}{NT}\sum_{i=1}^{n} F^{(r)}\lambda_{ir}\lambda_{ir}' F^{(r)}\hat{F}^{(r)} \\
&= \frac{1}{NT}\sum_{i=1}^{n} X_i^{(r)}\hat{\delta}_r\hat{\delta}_r' X_i^{(r)\prime}\hat{F}^{(r)} - \frac{1}{NT}\sum_{i=1}^{n} X_i^{(r)}\hat{\delta}_r\lambda_{ir}' F^{(r)'}\hat{F}^{(r)} - \frac{1}{NT}\sum_{i=1}^{n} X_i^{(r)}\hat{\delta}_r\varepsilon_i^{(r)\prime}\hat{F}^{(r)} \\
&\quad - \frac{1}{NT}\sum_{i=1}^{n} F^{(r)}\lambda_{ir}\hat{\delta}_r' X_i^{(r)\prime}\hat{F}^{(r)} - \frac{1}{NT}\sum_{i=1}^{n} \varepsilon_i^{(r)}\hat{\delta}_r' X_i^{(r)\prime}\hat{F}^{(r)} - \frac{1}{NT}\sum_{i=1}^{N} X_i^{(r)}\hat{\delta}_r'\Delta_i^{(r)'}\hat{F}^{(r)} - \frac{1}{NT}\sum_{i=1}^{N} \Delta_i^{(r)}\hat{\delta}_r' X_i^{(r)\prime}\hat{F}^{(r)} \\
&\quad + \frac{1}{NT}\sum_{i=1}^{n} F^{(r)}\lambda_{ir}\varepsilon_i^{(r)\prime}\hat{F}^{(r)} + \frac{1}{NT}\sum_{i=1}^{N} \varepsilon_i^{(r)}\lambda_{ir}' F^{(r)'}\hat{F}^{(r)} + \frac{1}{NT}\sum_{i=1}^{N} \varepsilon_i^{(r)}\varepsilon_i^{(r)\prime}\hat{F}^{(r)} + \frac{1}{NT}\sum_{i=1}^{n} F^{(r)}\lambda_{ir}\Delta_i^{(r)'}\hat{F}^{(r)} \\
&\quad + \frac{1}{NT}\sum_{i=1}^{n} \Delta_i^{(r)}\Delta_i^{(r)'}\hat{F}^{(r)} + \frac{1}{NT}\sum_{i=1}^{n} \varepsilon_i^{(r)}\Delta_i^{(r)'}\hat{F}^{(r)} + \frac{1}{NT}\sum_{i=1}^{n} \Delta_i^{(r)}\lambda_{ir}' F^{(r)'}\hat{F}^{(r)} + \frac{1}{NT}\sum_{i=1}^{n} \Delta_i^{(r)}\varepsilon_i^{(r)\prime}\hat{F}^{(r)} \\
&\equiv \sum_{s=1}^{15} I_s^{(r)}. \qquad \text{(C.3)}
\end{aligned}
$$

Pre-multiplying both sides of (C.3) by $\left(N^{-1}\Lambda_r'\Lambda_r\right)^{1/2} T^{-1}F^{(r)\prime}$, we have

$$
\left(\frac{\Lambda_r'\Lambda_r}{N}\right)^{1/2}\left(\frac{F^{(r)'}\hat{F}^{(r)}}{T}\right)\hat{V}_{NT}^{(r)} = \left(\frac{\Lambda_r'\Lambda_r}{N}\right)^{1/2}\left(\frac{F^{(r)'}F^{(r)}}{T}\right)\frac{\Lambda_r'\Lambda_r}{N}\left(\frac{F^{(r)'}\hat{F}^{(r)}}{T}\right) + d_{NT}^{(r)}, \qquad \text{(C.4)}
$$

where $d_{NT}^{(r)} \equiv (\frac{1}{N}\Lambda_r'\Lambda_r)^{1/2}\frac{1}{T}F^{(r)'}\sum_{s=1}^{15} I_s^{(r)}$. We note that Su and Wang (2017, 2020) have shown that $||(\frac{1}{N}\Lambda_r'\Lambda_r)^{1/2} \times \frac{1}{T}F^{(r)'}\sum_{s=8}^{15} I_s^{(r)}|| = O_P(T^{-1}h^{-1} + N^{-1/2} + h^2)$. This, along with the asymptotic nonsingularity of $\frac{\Lambda_r'\Lambda_r}{N}$, implies that it suffices to study the order of $\frac{1}{T}F^{(r)'}I_s^{(r)}$ for $s \in [7]$. Noting that $\frac{1}{T}\left\|\hat{F}^{(r)}\right\|^2 = \text{tr}(\frac{1}{T}\hat{F}^{(r)\prime}\hat{F}^{(r)}) = R$, we have by the triangle and CS inequalities,

$$
\left\|\frac{1}{T}F^{(r)'}I_1^{(r)}\right\| \le R^{1/2}\frac{1}{T^{1/2}}\left\|F^{(r)}\right\|\frac{1}{NT}\sum_{i=1}^{N}\|X_i^{(r)}\|^2\left\|\hat{\delta}_r\right\|^2 = O_P(\|\hat{\delta}_r\|^2).
$$

Similarly, we can readily show that $\left\|\frac{1}{T}F^{(r)'}I_l^{(r)}\right\| = O_P(\|\hat{\delta}_r\|)$ for $l = 2, 3, 4, 5$. Next,

$$
\left\|\frac{1}{T}F^{(r)'}I_6^{(r)}\right\| \le R^{1/2}\left[\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{F^{(r)'}X_i^{(r)}}{T}\right\|^2\right]^{1/2}\left[\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\Delta_i^{(r)}}{T}\right\|^2\right]^{1/2}\left\|\hat{\delta}_r\right\| = O_P(\|\hat{\delta}_r\|h)
$$

where we use the fact that $\frac{1}{NT}\sum_{i=1}^{N}\left\|\Delta_i^{(r)}\right\|^2 = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left\|\Delta_i^{(r)}(t,r)\right\|^2 = O_P(h^2)$. Similarly, we can show that $\left\|\frac{1}{T}F^{(r)'}I_7^{(r)}\right\| = O_P(\|\hat{\delta}_r\|h)$. In sum, we have shown that $d_{NT}^{(r)} = O_P(T^{-1}h^{-1} + N^{-1/2} + h^2) + O_P(\|\hat{\delta}_r\|)$.

Now, letting

$$
B_{NT}^{(r)} = \left(\frac{\Lambda_r'\Lambda_r}{N}\right)^{1/2}\left(\frac{F^{(r)'}F^{(r)}}{T}\right)\left(\frac{\Lambda_r'\Lambda_r}{N}\right)^{1/2} \text{ and } R_{NT}^{(r)} = \left(\frac{\Lambda_r'\Lambda_r}{N}\right)^{1/2}\left(\frac{F^{(r)'}\hat{F}^{(r)}}{T}\right),
$$

we can rewrite (C.4) as follows: $[B_{NT}^{(r)} + d_{NT}^{(r)}R_{NT}^{(r)-1}]R_{NT}^{(r)} = R_{NT}^{(r)}\hat{V}_{NT}^{(r)}$. Hence, each column of $R_{NT}^{(r)}$ is a non-standardized eigenvector of the matrix $B_{NT}^{(r)} + d_{NT}^{(r)}R_{NT}^{(r)-1}$. Let $\tilde{V}_{NT}^{(r)}$ be a diagonal matrix consisting of the diagonal elements of $R_{NT}^{(r)\prime}R_{NT}^{(r)}$. Denote the standardized eigenvector $\Upsilon_{NT}^{(r)}=R_{NT}^{(r)}\tilde{V}_{NT}^{(r)-1/2}$. Hence, we have $[B_{NT}^{(r)}+d_{NT}^{(r)}R_{NT}^{(r)-1}]\Upsilon_{NT}^{(r)}=\Upsilon_{NT}^{(r)}\hat{V}_{NT}^{(r)}$. That is, $\hat{V}_{NT}^{(r)}$ contains the eigenvalues of $B_{NT}^{(r)}+d_{NT}^{(r)}R_{NT}^{(r)-1}$ with the corresponding normalized eigenvectors contained in $\Upsilon_{NT}^{(r)}$. It is trivial to show that

$$\left\|B_{NT}^{(r)}+d_{NT}^{(r)}R_{NT}^{(r)-1}-B_r\right\|=O_P\left(C_{NT}^{-1}\right)+O_P(\|\hat{\delta}_r\|), \tag{C.5}$$

where B_r denotes the probability of $B_{NT}^{(r)}$, i.e., $B_r=\Sigma_{\Lambda_r}^{1/2}\Sigma_F\Sigma_{\Lambda_r}^{1/2}$. By the perturbation theory for eigenvalues of Hermitian matrices (e.g., Stewart and Sun (1990, p. 203)),

$$\left|\mu_j(B_{NT}^{(r)}+d_{NT}^{(r)}R_{NT}^{(r)-1})-\mu_j\left(B_r\right)\right|\leq\left\|B_{NT}^{(r)}+d_{NT}^{(r)}R_{NT}^{(r)-1}-B_r\right\|=O_P(C_{NT}^{-1})+O_P(\|\hat{\delta}_r\|),$$

where $\mu_j\left(A\right)$ denotes the jth largest eigenvalue of a symmetric matrix A and $j=1,...,R$. That is, $\left\|V_{NT}^{(r)}-V_r\right\|=O_P(C_{NT}^{-1})+O_P(\|\hat{\delta}_r\|)$.

(ii)-(iv) The proofs are essentially the same as those of Lemma A.1(ii)-(iv) in Su and Wang (2017) and thus omitted here.

(v) As in the proof of Proposition 3.1, we have $\left\|P_{\hat{F}^{(r)}}-P_{F^{(r)}}\right\|^2=2\text{tr}(\mathbb{I}_R-T^{-1}\hat{F}^{(r)\prime}P_{F^{(r)}}\hat{F}^{(r)})$. Using the decomposition that $\hat{F}^{(r)}=(\hat{F}^{(r)}-F^{(r)}H^{(r)}-B^{(r)})+F^{(r)}H^{(r)}+B^{(r)}$ and Lemma A.2 (iv) and Lemma A.3 (ii)-(iii) below, we can show that $\mathbb{I}_R-T^{-1}\hat{F}^{(r)\prime}P_{F^{(r)}}\hat{F}^{(r)}=\mathbb{I}_R-T^{-1}\hat{F}^{(r)\prime}F^{(r)}\left(F^{(r)\prime}F^{(r)}\right)^{-1}F^{(r)\prime}\hat{F}^{(r)}=O_P(C_{NT}^{-2})+O_P(\|\hat{\delta}_r\|)$. ■

Proof of Lemma A.3. (i) We consider the decomposition in (A.7). By CS inequality, we have

$$\frac{1}{T}\sum_{t=1}^{T}\left\|\hat{F}_t^{(r)}-H^{(r)\prime}F_t^{(r)}-B_t^{(r)}\right\|^2\leq\left\|\hat{V}_{NT}^{(r)-1}\right\|^2\frac{16}{T}\sum_{t=1}^{T}\sum_{l=1}^{16}\left\|\hat{V}_{NT}^{(r)}A_l(t,r)\right\|^2.$$

By Lemma A.2(i), we can bound $T^{-1}\sum_{t=1}^{T}\|A_l(t,r)\|^2$ by determining the probability order of $T^{-1}\sum_{t=1}^{T}\left\|\hat{V}_{NT}^{(r)}A_l(t,r)\right\|^2$. The terms $A_1(t,r)$ to $A_9(t,r)$ are almost the same as those defined in the proof of Theorem 2.1 of Su and Wang (2020), except that their bias term $D(s,r)F_s^{(r)}$ has been replaced by $\Delta_s^{(r)}=X_s^{(r)}L(s,r)+D(s,r)F_s^{(r)}$ here. Following the proof of Lemma A.2 in Su and Wang (2017, 2020), we can show that $\sum_{l=1}^{9}T^{-1}\sum_{t=1}^{T}\left\|\hat{V}_{NT}^{(r)}A_l(t,r)\right\|^2=O_P(C_{NT}^{-2})$. Similar to the proof of Lemma A.2(i), it is easy to show that $\sum_{l=10}^{15}T^{-1}\sum_{t=1}^{T}\left\|\hat{V}_{NT}^{(r)}A_l(t,r)\right\|^2=O_P(\|\hat{\delta}_r\|)$. Then $T^{-1}\sum_{t=1}^{T}\left\|\hat{F}_t^{(r)}-H^{(r)\prime}F_t^{(r)}-B_t^{(r)}\right\|^2=O_P(C_{NT}^{-2})+O_P(\|\hat{\delta}_r\|)$.

(ii) By (A.7), we obtain $\frac{1}{T}(\hat{F}^{(r)}-F^{(r)}H^{(r)}-B^{(r)})'F^{(r)}H^{(r)}=\hat{V}_{NT}^{(r)-1}\sum_{l=1}^{16}\bar{A}_l(r)H^{(r)}$, where $\bar{A}_l(r)=\frac{1}{T}\sum_{t=1}^{T}\hat{V}_{NT}^{(r)}A_l(t,r)F_t^{(r)\prime}$. Su and Wang (2017, 2020) have studied the terms $\bar{A}_l(r)$, $l\in[9]$. Following the results there, we have $\sum_{l=1}^{9}\bar{A}_l(r)=O_P(C_{NT}^{-2})$. For $\bar{A}_{10}(r)$ and $\bar{A}_{11}(r)$, we have

$$\begin{aligned}\left\|\bar{A}_{10}(r)\right\| &\leq R^{1/2}T^{-1/2}\left\|F^{(r)}\right\|\frac{1}{NT}\sum_{i=1}^{N}\|X_i^{(r)}\|^2\|\hat{\delta}_r\|^2=O_P(\|\hat{\delta}_r\|^2),\text{ and}\\ \left\|\bar{A}_{11}(r)\right\| &\leq R^{1/2}\left\{\frac{1}{N^2T}\sum_{s=1}^{T}\left\|X_s^{(r)\prime}\Lambda_r\right\|^2\right\}^{1/2}\frac{1}{T}\sum_{t=1}^{T}\left\|F_t^{(r)}\right\|^2\|\hat{\delta}_r\|=O_P(\|\hat{\delta}_r\|).\end{aligned}$$

Similarly, we can readily show that $\left\|\bar{A}_l(r)\right\|=O_P(\|\hat{\delta}_r\|)$ for $l=12$, 13, and 14, and $\left\|\bar{A}_{15}(r)\right\|=O_P(\|\hat{\delta}_r\|h)$ for $l=15$ and 16. Combining the above results and using Lemma A.2(i) yield the claim in part (ii) of the lemma.

(iii) This follows from (i) and (ii) and the triangle inequality.

(iv)-(v) The proof is almost the same as part (ii) of the lemma and omitted for brevity. ■

Proof of Lemma A.4. The proof is similar to that of Lemma A.3 and we only sketch it here.

(i) Let $\Upsilon^{(r)} = \hat{F}^{(r)} - F^{(r)}H^{(r)} - B^{(r)}$. Then

$$\begin{aligned}
& (NT)^{-1}\sum_{i=1}^{N} X_i^{(r)\prime} M_{\hat{F}^{(r)}} \Upsilon^{(r)} \lambda_{ir}' \\
= \; & (NT)^{-1}\sum_{i=1}^{N} X_i^{(r)\prime} \Upsilon^{(r)} \lambda_{ir}' - (NT)^{-1}\sum_{i=1}^{N} X_i^{(r)\prime} P_{F^{(r)}} \Upsilon^{(r)} \lambda_{ir}' - (NT)^{-1}\sum_{i=1}^{N} X_i^{(r)\prime} (P_{\hat{F}^{(r)}} - P_{F^{(r)}}) \Upsilon^{(r)} \lambda_{ir}' \\
\equiv \; & II_1 - II_2 - II_3.
\end{aligned}$$

Following the proof of Lemma A.3(ii), we can show that $\|II_l\| = O_P(C_{NT}^{-2}) + O_P(\|\hat{\delta}_r\|)$ for $l = 1, 2$. For II_3, we can readily apply Lemmas A.2(v) and A.3(i) to obtain

$$\|II_3\| \leq \left\{ N^{-1}T^{-1/2} \sum_{i=1}^{N} \left\| X_i^{(r)} \right\| \|\lambda_{ir}\| \right\} \left\| P_{\hat{F}^{(r)}} - P_{F^{(r)}} \right\| \frac{1}{T^{1/2}} \left\| \Upsilon^{(r)} \right\| = O_P(C_{NT}^{-2}) + O_P(\|\hat{\delta}_r\|).$$

It follows that $(NT)^{-1} \sum_{i=1}^{N} X_i^{(r)\prime} M_{\hat{F}^{(r)}} \Upsilon^{(r)} \lambda_{ir}' = O_P(C_{NT}^{-2}) + O_P(\|\hat{\delta}_r\|)$.

(ii) Let ω be an arbitrary nonrandom P-vector with $\|\omega\| = 1$. Note that $\frac{1}{NT}\sum_{i=1}^{N} \lambda_{ir}\omega' X_i^{(r)\prime} M_{\hat{F}^{(r)}} B^{(r)} = \frac{1}{NT}\sum_{i=1}^{N} \lambda_{ir}\omega' X_i^{(r)\prime} B^{(r)} - \frac{1}{NT}\sum_{i=1}^{N} \lambda_{ir}\omega' X_i^{(r)\prime} P_{F^{(r)}} B^{(r)} - \frac{1}{NT}\sum_{i=1}^{N} \lambda_{ir}\omega' X_i^{(r)\prime} \left(P_{\hat{F}^{(r)}} - P_{F^{(r)}}\right) B^{(r)} \equiv II_4 - II_5 - II_6$. Recall that $B_t^{(r)} = k_{h,tr}^{1/2} C_{1t}^{(r)} \frac{t-r}{T} + k_{h,tr}^{1/2} C_{2t}^{(r)} (\frac{t-r}{T})^2 + k_{h,tr}^{1/2} C_{3:4,t}^{(r)} h^2 \kappa_2$ where $C_{3:4,t}^{(r)} = C_{3t}^{(r)} + C_{4t}^{(r)}$, we can readily show that

$$\begin{aligned}
II_4 &= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} \lambda_{ir}\omega' X_{it}^{(r)} B_t^{(r)\prime} = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} k_{h,tr}^{*} \lambda_{ir}\omega' X_{it} \left[C_{1t}^{(r)} \frac{t-r}{T} + C_{2t}^{(r)} (\frac{t-r}{T})^2 + C_{3:4,t}^{(r)} h^2 \kappa_2 \right]' \\
&= \overline{II}_4 + O_P\left(h^2\right),
\end{aligned}$$

where $\overline{II}_4 = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} k_{h,tr}^{*} \lambda_{ir} X_{it}' \omega C_{1t}^{(r)\prime} \frac{t-r}{T}$. Noting that $C_{1t}^{(r)} = \hat{V}_{NT}^{(r)-1} H^{(r)\prime} \Sigma_F \left(\Lambda_r' A_{1,tr}/N\right)$ and $A_{1,tr} = X_t \beta_r^{(1)} + \Lambda_r^{(1)} F_t$, it is easy to see that the probability order of $\overline{II}_4$ is determined by that of $\overline{II}_{4,1}$ and $\overline{II}_{4,2}$, where

$$\begin{aligned}
\overline{II}_{4,1} &= \frac{1}{NT}\sum_{t=1}^{T} k_{h,tr}^{*} \left(\frac{1}{N}\sum_{i=1}^{N} \lambda_{ir} X_{it}' \right) \omega \beta_r^{(1)\prime} X_t' \Lambda_r \frac{t-r}{T} \text{ and} \\
\overline{II}_{4,2} &= \frac{1}{NT}\sum_{t=1}^{T} k_{h,tr}^{*} \left(\frac{1}{N}\sum_{i=1}^{N} \lambda_{ir} X_{it}' \right) \omega F_t' \Lambda_r^{(1)} \frac{t-r}{T}.
\end{aligned}$$

Using the fact that $\frac{1}{N}\sum_{i=1}^{N} \lambda_{ir} X_{it}' = \frac{1}{N}\sum_{i=1}^{N} \lambda_{ir} E(X_{i1}') + O_P((N/\ln T)^{-1/2})$ uniformly in t, we can show that

$$\overline{II}_{4,1} = \frac{1}{N}\sum_{i=1}^{N} \lambda_{ir} E(X_{i1}') \omega \beta_r^{(1)\prime} \frac{1}{N}\sum_{j=1}^{N} \frac{1}{T}\sum_{t=1}^{T} k_{h,tr}^{*} X_{jt} \lambda_{jr}' \frac{t-r}{T} + O_P((N/\ln T)^{-1/2} h).$$

This, in conjunction with the fact that

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T} k_{h,tr}^{*} X_{jt} \frac{t-r}{T} &= \frac{1}{T}\sum_{t=1}^{T} k_{h,tr}^{*} E(X_{j1}) \frac{t-r}{T} + \frac{1}{T}\sum_{t=1}^{T} k_{h,tr}^{*} [X_{jt} - E\left(X_{jt}\right)] \frac{t-r}{T} \\
&= O(T^{-1}) + O_P((Th/\ln T)^{-1/2}\, h) \text{ uniformly in } j,
\end{aligned}$$

implies that $\overline{II}_{4,1} = O_P((N/\ln T)^{-1/2} h + (Th/\ln T)^{-1/2}\, h + T^{-1})$. Similarly, $\overline{II}_{4,2} = O_P((N/\ln T)^{-1/2} h +$

$(Th/\ln T)^{-1/2}h+T^{-1})$. Thus $\overline{II}_4=O_P((N/\ln T)^{-1/2}h+(Th/\ln T)^{-1/2}h+T^{-1})$ and

$$II_4=O_P((N/\ln T)^{-1/2}h+(Th/\ln T)^{-1/2}h+T^{-1})+O_P\left(h^2\right)=O_P\left(h^2\right).$$

Similarly, we can show that $II_5=O_P\left(h^2\right)$. Following the analysis of II_3 in the proof of (ii), we can show that

$$\begin{aligned}\|II_6\| &= \left\|\frac{1}{NT}\sum_{i=1}^{N}\lambda_{ir}\omega' X_i^{(r)\prime}\left(P_{\hat{F}^{(r)}}-P_{F^{(r)}}\right)B^{(r)}\right\| \lesssim \left\|P_{\hat{F}^{(r)}}-P_{F^{(r)}}\right\|\frac{1}{NT^{1/2}}\sum_{i=1}^{N}\left\|X_i^{(r)}\right\|\frac{1}{T^{1/2}}\left\|B^{(r)}\right\| \\ &= O_P((C_{NT}^{-1}+\|\hat{\delta}_r\|^{1/2})h).\end{aligned}$$

It follows that $\frac{1}{NT}\sum_{i=1}^{N}\lambda_{ir}\omega' X_i^{(r)\prime}M_{\hat{F}^{(r)}}B^{(r)}=O_P(h^2)+O_P(\|\hat{\delta}_r\|)$ and the conclusion follows.

(iii) $N^{-1}T^{-1}\sum_{j=1}^{N}\lambda_{jr}\varepsilon_j^{(r)\prime}\hat{F}^{(r)}=N^{-1}T^{-1}\sum_{j=1}^{N}\lambda_{jr}\varepsilon_j^{(r)\prime}\Upsilon^{(r)}+N^{-1}T^{-1}\sum_{j=1}^{N}\lambda_{jr}\varepsilon_j^{(r)\prime}F^{(r)}H^{(r)}+N^{-1}T^{-1}\sum_{j=1}^{N}$ $\lambda_{jr}\varepsilon_j^{(r)\prime}B^{(r)}\equiv II_7+II_8+II_9$. Following the proof of Lemma A.3(ii), we can show that $II_7=O_P(C_{NT}^{-2})+O_P(\|\hat{\delta}_r\|)$. For II_8, we have $N^{-1}T^{-1}\sum_{j=1}^{N}\lambda_{jr}\varepsilon_j^{(r)\prime}F^{(r)}=N^{-1}T^{-1}\sum_{j=1}^{N}\sum_{t=1}^{T}k_{tr}^{*}\lambda_{jr}\varepsilon_{jt}F_t=O_P((NTh)^{-1/2})$, implying $II_8=O_P((NTh)^{-1/2})$. Similarly, $II_9=O_P((NTh)^{-1/2}h)$. Consequently,

$$N^{-1}T^{-1}\sum_{j=1}^{N}\lambda_{jr}\varepsilon_j^{(r)\prime}\hat{F}^{(r)}=O_P(C_{NT}^{-2})+O_P(\|\hat{\delta}_r\|)+O_P((NTh)^{-1/2})=O_P(C_{NT}^{-2})+O_P(\|\hat{\delta}_r\|).$$

(iv) The result is related to Lemma A.4 of Bai (2009). Following his arguments, one expect that $\frac{1}{T}\Upsilon^{(r)\prime}\varepsilon_i^{(r)}=(Th)^{-1/2}O_P(\|\hat{\delta}_r\|)+O_P(C_{NT}^{-2})$ for each i. Here we prove the corresponding mean square error result. By (A.7),

$$\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\Upsilon^{(r)\prime}\varepsilon_i^{(r)}\right\|^2=\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{t=1}^{T}\Upsilon_t^{(r)}\varepsilon_{it}^{(r)}\right\|^2\leq 16\left\|\hat{V}_{NT}^{(r)-1}\right\|\sum_{l=1}^{16}A_l^{(r)},$$

where $A_l^{(r)}=\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{t=1}^{T}\hat{V}_{NT}^{(r)}A_l\left(t,r\right)\varepsilon_{it}^{(r)}\right\|^2$ for $l\in[16]$. Following the proof of Lemma A.4 of Bai (2009) and the proof of Lemma A.3, and using the results in Lemmas A.2-A.3, we can readily show that $\frac{1}{T}\sum_{t=1}^{T}\hat{V}_{NT}^{(r)}A_l\left(t,r\right)\varepsilon_{it}^{(r)}=O_P(C_{NT}^{-4}+\|\hat{\delta}_r\|^2)$ for $l\in[9]$. In addition, it is trivial to show that $\frac{1}{T}\sum_{t=1}^{T}\hat{V}_{NT}^{(r)}$ $A_l\left(t,r\right)\varepsilon_{it}^{(r)}=O_P(\|\hat{\delta}_r\|^2)$. Then $\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\Upsilon^{(r)\prime}\varepsilon_i^{(r)}\right\|^2=O_P(C_{NT}^{-4}+\|\hat{\delta}_r\|^2)$.

(v) By (A.7) we have

$$\begin{aligned}&\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\frac{1}{T}X_i^{(r)\prime}F^{(r)}[\frac{1}{T}F^{(r)\prime}F^{(r)}]^{-1}[\hat{F}^{(r)}H^{(r)-1}-F^{(r)}-B^{(r)}H^{(r)-1}]'\varepsilon_i^{(r)} \\ =\;&\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\frac{1}{T}X_i^{(r)\prime}F^{(r)}[\frac{1}{T}F^{(r)\prime}F^{(r)}]^{-1}\left(H^{(r)-1}\right)'\Upsilon^{(r)\prime}\varepsilon_i^{(r)} \\ =\;&\sum_{l=1}^{15}\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\frac{1}{T}X_i^{(r)\prime}F^{(r)}[\frac{1}{T}F^{(r)\prime}F^{(r)}]^{-1}\left(H^{(r)-1}\right)'A_l(t,r)\varepsilon_{it}^{(r)}\equiv\sum_{l=1}^{15}\bar{A}_l^{(r)}.\end{aligned}$$

Let ω be any nonrandom P-vector with $\|\omega\|=1$. Then

$$\begin{aligned}\left|\omega'\bar{A}_1^{(r)}\right| &= \left|\sqrt{Nh}\mathrm{tr}\left\{\frac{F^{(r)}}{T^{1/2}}\left(\frac{F^{(r)\prime}F^{(r)}}{T}\right)^{-1}\left(H^{(r)-1}\right)'\hat{V}_{NT}^{(r)-1}\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\hat{F}_s^{(r)}E(\varepsilon_s^{(r)\prime}\varepsilon_t^{(r)}/N)\varepsilon_{it}^{(r)}\omega' X_i^{(r)\prime}\right\}\right| \\ &\lesssim \sqrt{Nh}\left\|\frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\hat{F}_s^{(r)}E(\varepsilon_s^{(r)\prime}\varepsilon_t^{(r)}/N)\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{it}^{(r)}\omega' X_i^{(r)\prime}\right\|\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sqrt{Nh}\left\|\frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}F_s^{(r)}E(\varepsilon_s^{(r)\prime}\varepsilon_t^{(r)}/N)\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{it}^{(r)}\omega' X_i^{(r)\prime}\right\| \\
&\quad +\sqrt{Nh}\left\|\frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}B_s^{(r)}E(\varepsilon_s^{(r)\prime}\varepsilon_t^{(r)}/N)\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{it}^{(r)}\omega' X_i^{(r)\prime}\right\| \\
&\quad +\sqrt{Nh}\left\|\frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}(\hat{F}_s^{(r)}-H^{(r)\prime}F_s^{(r)}-B_s^{(r)})E(\varepsilon_s^{(r)\prime}\varepsilon_t^{(r)}/N)\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{it}^{(r)}\omega' X_i^{(r)\prime}\right\| \\
&\equiv II_{10}+II_{11}+II_{12}.
\end{aligned}
$$

As in Su and Wang (2017), we can readily show that $\frac{1}{T^2}\sum_{t=1}^{T}||\sum_{s=1}^{T}F_s^{(r)}E(\varepsilon_s^{(r)\prime}\varepsilon_t^{(r)}/N)||^2=\frac{1}{T}$ and $\frac{1}{T^2}\sum_{t=1}^{T}||\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{it}^{(r)}\omega' X_i^{(r)\prime}||^2=O_P\left((Th)^{-1}+N^{-1}\right)$ by Markov inequality under Assumptions A.1(vi) and A.2(iii) as

$$
\begin{aligned}
&\frac{1}{T^2}\sum_{t=1}^{T}E\left\|\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{it}^{(r)}\omega' X_i^{(r)\prime}\right\|^2 \\
&= \frac{1}{N^2T^2}\sum_{i,j=1}^{N}\sum_{s,t=1}^{T}k_{h,tr}^{*}k_{h,sr}^{*}E\{[\varepsilon_{it}\varepsilon_{jt}-E(\varepsilon_{it}\varepsilon_{jt})]X_{is}'X_{js}\}+\frac{1}{N^2T^2}\sum_{i,j=1}^{N}\sum_{s,t=1}^{T}k_{h,tr}^{*}k_{h,sr}^{*}E(\varepsilon_{it}\varepsilon_{jt})E(X_{is}'X_{js}) \\
&\lesssim h^{-1}\max_{i,j,s}\sum_{s=1}^{T}|E\{[\varepsilon_{it}\varepsilon_{jt}-E(\varepsilon_{it}\varepsilon_{jt})]X_{is}'X_{js}\}|\frac{1}{T^2}\sum_{t=1}^{T}k_{h,tr}^{*}+\max_t\frac{1}{N^2}\sum_{i,j=1}^{N}|E(\varepsilon_{it}\varepsilon_{jt})|\frac{1}{T^2}\sum_{s,t=1}^{T}k_{h,tr}^{*}k_{h,sr}^{*} \\
&= O\left((Th)^{-1}+N^{-1}\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
II_{10} &\leq \sqrt{Nh}\left\{\frac{1}{T^2}\sum_{t=1}^{T}\left\|\sum_{s=1}^{T}F_s^{(r)}E(\varepsilon_s^{(r)\prime}\varepsilon_t^{(r)}/N)\right\|^2\right\}^{1/2}\left\{\frac{1}{T^2}\sum_{t=1}^{T}\left\|\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{it}^{(r)}\omega' X_i^{(r)\prime}\right\|^2\right\}^{1/2} \\
&= \sqrt{Nh}O_P(T^{-1/2})O_P((Th)^{-1/2}+N^{-1/2})=o_P(1).
\end{aligned}
$$

Similarly, $II_{11}=o_P(1)$. For II_{12}, we have by CS equality and Lemma A.3(i),

$$
\begin{aligned}
II_{12} &\leq \sqrt{Nh}\left\|\frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}(\hat{F}_s^{(r)}-H^{(r)\prime}F_s^{(r)}-B_s^{(r)})E(\varepsilon_s^{(r)\prime}\varepsilon_t^{(r)}/N)\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{it}^{(r)}\omega' X_i^{(r)\prime}\right\| \\
&\leq \left\{\frac{1}{T}\sum_{s=1}^{T}\left\|\hat{F}_s^{(r)}-H^{(r)\prime}F_s^{(r)}-B_s^{(r)}\right\|^2\right\}^{1/2}\max_t\left\{\left\|\sum_{s=1}^{T}E(\varepsilon_s^{(r)\prime}\varepsilon_t^{(r)}/N)\right\|^2\right\}^{1/2} \\
&\quad \times\left\|\frac{\sqrt{Nh}}{NT^{3/2}}\sum_{t=1}^{T}\sum_{i=1}^{N}\varepsilon_{it}^{(r)}\omega' X_i^{(r)\prime}\right\| \\
&= O_P(\|\hat{\delta}_r\|^{1/2}+C_{NT}^{-1})O(1)O_P(1)=o_P(1).
\end{aligned}
$$

Then $\bar{A}_1^{(r)}=o_P(1)$. Similarly, we can show that $\bar{A}_l^{(r)}=o_P(1)$ for $l=2,4,5,...,9$. For $\bar{A}_3^{(r)}$, in view of the fact that $(H^{(r)-1})'\hat{V}_{NT}^{(r)-1}=\left(N^{-1}\Lambda_r'\Lambda_r\right)^{-1}(T^{-1}\hat{F}^{(r)\prime}F^{(r)})^{-1}$ by the definition of $H^{(r)}$, we have

$$
\bar{A}_3^{(r)}=\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\frac{X_i^{(r)\prime}F^{(r)}}{T}\left(\frac{F^{(r)\prime}F^{(r)}}{T}\right)^{-1}\left(\frac{\Lambda_r'\Lambda_r}{N}\right)^{-1}\Lambda_r'\varepsilon_t^{(r)}/N\varepsilon_{it}^{(r)}=\sqrt{\frac{Th}{N}}\psi_{NT}^{(r)}
$$

where $\psi_{NT}^{(r)}=\frac{1}{N}\sum_{i=1}^{N}\sum_{k=1}^{N}\frac{X_i^{(r)\prime}F^{(r)}}{T}\left(\frac{F^{(r)\prime}F^{(r)}}{T}\right)^{-1}\left(\frac{\Lambda_r'\Lambda_r}{N}\right)^{-1}\lambda_{kr}\left(\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}^{(r)}\varepsilon_{kt}^{(r)}\right)$.

Next, it is easy to obtain the rough bound for the remainder terms:

$$\begin{aligned}\bar{A}_{10}^{(r)} &= \sqrt{NTh}O_P(\|\hat{\delta}_r\|^2\,(Nh)^{-1/2})),\ \ \bar{A}_l^{(r)} = \sqrt{NTh}O_P(\|\hat{\delta}_r\|\,(Nh)^{-1/2}) \text{ for } l = 11,\ 13,\ 14 \text{ and } 16, \text{ and} \\ \bar{A}_l^{(r)} &= \sqrt{NTh}O_P(\|\hat{\delta}_r\|h\,(Nh)^{-1/2}) \text{ for } l = 12,\ 15.\end{aligned}$$

In sum, we have $\frac{\sqrt{NTh}}{NT}\sum_{i=1}^N \frac{X_i^{(r)\prime}F^{(r)}}{T}\left(\frac{F^{(r)\prime}F^{(r)}}{T}\right)^{-1}[\hat{F}^{(r)}H^{(r)-1}-F^{(r)}-B^{(r)}H^{(r)-1}]'\varepsilon_i^{(r)} = \sqrt{\frac{Th}{N}}\psi_{NT}^{(r)}+\sqrt{T}O_P(\|\hat{\delta}_r\|)+o_P(1)$. ■

Proof of Lemma A.5. Recall that $H^{(r)} = (N^{-1}\Lambda_r'\Lambda_r)(T^{-1}F^{(r)\prime}\hat{F}^{(r)})\hat{V}_{NT}^{(r)-1}$. By (C.3), $\hat{F}^{(r)}H^{(r)-1} - F^{(r)} = \sum_{l=1}^{15} I_l^{(r)}G^{(r)}$, where $G^{(r)} = (\frac{1}{T}F^{(r)\prime}\hat{F}^{(r)})^{-1}(\frac{1}{N}\Lambda_r'\Lambda_r)^{-1}$ satisfies $\left\|G^{(r)}\right\| = O_P(1)$ as $(T^{-1}\hat{F}^{(r)\prime}F^{(r)})^{-1} = Q_r^{-1} + o_P(1)$ by Lemma A.2(ii) and $N^{-1}\Lambda_r'\Lambda_r = \Sigma_{\Lambda_r} + O(N^{-1/2})$ by Assumption 3(i). It follows that

$$\begin{aligned}\frac{1}{NTh}\sum_{i=1}^N X_i^{(r)\prime}M_{\hat{F}^{(r)}}F^{(r)}\lambda_{ir} &= \frac{1}{NT}\sum_{i=1}^N X_i^{(r)\prime}M_{\hat{F}^{(r)}}[F^{(r)} - \hat{F}^{(r)}H^{(r)-1}]\lambda_{ir} \\ &= -\sum_{l=1}^{15}\frac{1}{NT}\sum_{i=1}^N X_i^{(r)\prime}M_{\hat{F}^{(r)}}I_l^{(r)}G^{(r)}\lambda_{ir} \equiv \sum_{l=1}^{15} J_l^{(r)}.\end{aligned}$$

For $J_1^{(r)}$, we have

$$\left\|J_1^{(r)}\right\| = \left\|\frac{1}{NT}\sum_{i=1}^N X_i^{(r)\prime}M_{\hat{F}^{(r)}}I_1^{(r)}G^{(r)}\lambda_{ir}\right\| \lesssim \frac{1}{NT^{1/2}}\sum_{i=1}^N\left\|X_i^{(r)}\right\|T^{-1/2}\left\|I_1^{(r)}\right\|\max_i\|\lambda_{ir}\| = O_P(\|\hat{\delta}_r\|^2),$$

where we use the fact that $G^{(r)} = O_P(1)$, $\left\|M_{\hat{F}^{(r)}}\right\|_{\text{sp}} = 1$, $\frac{1}{NT^{1/2}}\sum_{i=1}^N\left\|X_i^{(r)}\right\| = O_P(1)$, $T^{-1/2}\left\|I_1^{(r)}\right\| = O_P(\|\hat{\delta}_r\|^2)$, and $\max_i\|\lambda_{ir}\| = O(1)$. For $J_2^{(r)}$, we have

$$J_2^{(r)} = \frac{1}{N^2T}\sum_{i=1}^N\sum_{j=1}^n X_i^{(r)\prime}M_{\hat{F}^{(r)}}X_j^{(r)}\hat{\delta}_r\lambda_{jr}'(\frac{1}{N}\Lambda_r'\Lambda_r)^{-1}\lambda_{ir} = \frac{1}{N^2T}\sum_{i=1}^N\sum_{j=1}^N X_i^{(r)\prime}M_{\hat{F}^{(r)}}X_j^{(r)}a_{ji}^{(r)}\hat{\delta}_r,$$

where $a_{ji}^{(r)} \equiv \lambda_{jr}'\left(\frac{1}{N}\Lambda_r'\Lambda_r\right)^{-1}\lambda_{ir}$. For $J_3^{(r)}$, we have

$$\begin{aligned}J_3^{(r)} &= \frac{1}{N^2T^2}\sum_{i=1}^N\sum_{j=1}^N X_i^{(r)\prime}M_{\hat{F}^{(r)}}X_j^{(r)}\hat{\delta}_r\varepsilon_j^{(r)\prime}\hat{F}^{(r)}G^{(r)}\lambda_{ir} \\ &= \frac{1}{N^2T^2}\sum_{i=1}^N\sum_{j=1}^N X_i^{(r)\prime}M_{\hat{F}^{(r)}}X_j^{(r)}\hat{\delta}_r\sum_{t=1}^T[\Upsilon_t^{(r)} + H^{(r)\prime}F_t^{(r)} + B_t^{(r)}]\varepsilon_{jt}^{(r)}G^{(r)}\lambda_{ir} \equiv \sum_{l=1}^3 J_{3l}^{(r)},\end{aligned}$$

where $\Upsilon_t^{(r)} \equiv \hat{F}_t^{(r)} - H^{(r)\prime}F_t^{(r)} - B_t^{(r)}$, $J_{31}^{(r)} = \frac{1}{N^2T^2}\sum_{i=1}^N\sum_{j=1}^N X_i^{(r)\prime}M_{\hat{F}^{(r)}}X_j^{(r)}\hat{\delta}_r\sum_{t=1}^T\Upsilon_t^{(r)}\varepsilon_{jt}^{(r)}G^{(r)}\lambda_{ir}$, and $J_{32}^{(r)}$ and $J_{33}^{(r)}$ are similarly defined. Note that by Lemma A.3(i)

$$\begin{aligned}\left\|J_{31}^{(r)}\right\| &= \left\|\frac{1}{N^2T^2}\sum_{i=1}^N\sum_{j=1}^N X_i^{(r)\prime}M_{\hat{F}^{(r)}}X_j^{(r)}\hat{\delta}_r\sum_{t=1}^T\Upsilon_t^{(r)}\varepsilon_{jt}^{(r)}G^{(r)}\lambda_{ir}\right\| \\ &\lesssim \left\{\frac{1}{T}\sum_{t=1}^T\left\|\Upsilon_t^{(r)}\right\|^2\right\}^{1/2}\max_j\left\{\frac{1}{T}\sum_{t=1}^T\left\|\varepsilon_{jt}^{(r)}\right\|^2\right\}^{1/2}\frac{1}{NT}\sum_{i=1}^N\left\|X_i^{(r)}\right\|^2\|\hat{\delta}_r\| \\ &= O_P(C_{NT}^{-1} + \|\hat{\delta}_r\|^{1/2})O_p(\|\hat{\delta}_r\|).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\left\|J_{32}^{(r)}\right\| &\lesssim \left\{\frac{1}{N}\sum_{j=1}^{N}\left\|\frac{1}{T}\sum_{t=1}^{T}F_t^{(r)}\varepsilon_{jt}^{(r)}\right\|^2\right\}^{1/2}\|\hat{\delta}_r\| = O_P((Th)^{-1/2})\|\hat{\delta}_r\| \text{ and} \\ \left\|J_{33}^{(r)}\right\| &\lesssim \left\{\frac{1}{N}\sum_{j=1}^{N}\left\|\frac{1}{T}\sum_{t=1}^{T}B_t^{(r)}\varepsilon_{jt}^{(r)}\right\|^2\right\}^{1/2}\|\hat{\delta}_r\| = O_P((Th)^{-1/2}\,h)\|\hat{\delta}_r\|\end{aligned}$$

by straightforward moment calculations and Markov inequality. It follows that $J_3^{(r)} = O_P(C_{NT}^{-1}+\|\hat{\delta}_r\|^{1/2})O_P(\|\hat{\delta}_r\|)$. Next, we consider J_4 term:

$$\begin{aligned}J_4^{(r)} &= \frac{1}{NT}\sum_{i=1}^{N}X_i^{(r)'}M_{\hat{F}^{(r)}}\frac{1}{NT}\sum_{j=1}^{N}F^{(r)}\lambda_{jr}\hat{\delta}_r'X_j^{(r)\prime}\hat{F}^{(r)}G^{(r)}\lambda_{ir} \\ &= \frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{j=1}^{N}X_i^{(r)'}M_{\hat{F}^{(r)}}[F^{(r)}H^{(r)} - \hat{F}^{(r)} + B^{(r)}]H^{(r)-1}\lambda\hat{\delta}_r'X_j^{(r)\prime}\hat{F}^{(r)}G^{(r)}\lambda_{ir} \\ &\quad - \frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{j=1}^{N}X_i^{(r)'}M_{\hat{F}^{(r)}}B^{(r)}H^{(r)-1}\lambda_{jr}\hat{\delta}_r'X_j^{(r)\prime}\hat{F}^{(r)}G^{(r)}\lambda_{ir} \equiv J_{41}^{(r)} - J_{42}^{(r)}.\end{aligned}$$

Noting that $T^{-1/2}\left\|F^{(r)} - \hat{F}^{(r)}H^{(r)-1} + B^{(r)}\right\| = O_P(C_{NT}^{-1}) + O_P(\|\hat{\delta}_r\|^{1/2})$ and $\|B_t^{(r)}\| = O_P(h)$ when $|t-r| \lesssim h$, one can readily show that $J_4^{(r)} = O_P(C_{NT}^{-1} + \|\hat{\delta}_r\|^{1/2} + h)O_P(\|\hat{\delta}_r\|)$.

For $J_5^{(r)}$, we have

$$\begin{aligned}\left|\omega' J_5^{(r)}\right| &= \left|\frac{1}{NT}\sum_{i=1}^{N}\omega' X_i^{(r)'}M_{\hat{F}^{(r)}}\frac{1}{NT}\sum_{j=1}^{N}\varepsilon_j^{(r)}\hat{\delta}_r'X_j^{(r)\prime}\hat{F}^{(r)}G^{(r)}\lambda_{ir}\right| \\ &= \left|\operatorname{tr}\left(\hat{\delta}_r'\frac{1}{NT}\sum_{j=1}^{N}X_j^{(r)\prime}\hat{F}^{(r)}G^{(r)}\frac{1}{NT}\sum_{i=1}^{N}\lambda_{ir}\omega' X_i^{(r)'}M_{\hat{F}^{(r)}}\varepsilon_j^{(r)}\right)\right| \lesssim \left\{\frac{1}{NT}\sum_{j=1}^{N}\left\|X_j^{(r)}\right\|^2\right\}^{1/2}\left\{\bar{J}_5^{(r)}\right\}^{1/2}\|\hat{\delta}_r\|,\end{aligned}$$

where $\bar{J}_5^{(r)} = \frac{1}{N}\sum_{j=1}^{N}\left\|\frac{1}{NT}\sum_{i=1}^{N}\lambda_{ir}\omega' X_i^{(r)'}M_{\hat{F}^{(r)}}\varepsilon_j^{(r)}\right\|^2$. For $\bar{J}_5^{(r)}$, we have

$$\begin{aligned}&\bar{J}_5^{(r)} \\ \leq\ &\frac{3}{N}\sum_{j=1}^{N}\left\{\left\|\frac{1}{NT}\sum_{i=1}^{N}\lambda_{ir}\omega' X_i^{(r)'}\varepsilon_j^{(r)}\right\|^2 + \left\|\frac{1}{NT}\sum_{i=1}^{N}\lambda_{ir}\omega' X_i^{(r)'}P_{F^{(r)}}\varepsilon_j^{(r)}\right\|^2 + \left\|\frac{1}{NT}\sum_{i=1}^{N}\lambda_{ir}\omega' X_i^{(r)'}(P_{\hat{F}^{(r)}} - P_{F^{(r)}})\varepsilon_j^{(r)}\right\|^2\right\} \\ \equiv\ &3\left(II_{10} + II_{11} + II_{12}\right).\end{aligned}$$

By moment calculations, we can show that $II_l = O_P\left((Th)^{-1}\right)$ for $l = 10, 11$. For II_{12}, we can apply Lemma A.2(v) to obtain $II_{12} = O_P(C_{NT}^{-2} + \|\hat{\delta}_r\|)$. It follows that $J_5^{(r)} = O_P(C_{NT}^{-1} + \|\hat{\delta}_r\|^{1/2})O_P(\|\hat{\delta}_r\|)$. Next,

$$\begin{aligned}J_6^{(r)} &= \frac{1}{NT}\sum_{i=1}^{N}X_i^{(r)'}M_{\hat{F}^{(r)}}\frac{1}{NT}\sum_{j=1}^{N}X_j^{(r)}\hat{\delta}_r\Delta_j^{(r)\prime}\hat{F}^{(r)}G^{(r)}\lambda_{ir} \\ &= \frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{j=1}^{N}X_i^{(r)'}M_{\hat{F}^{(r)}}X_j^{(r)}\hat{\delta}_r\sum_{t=1}^{T}\left\{(\beta_t - \beta_r)' X_{jt}^{(r)} + (\lambda_{jt} - \lambda_{jr})' F_t^{(r)}\right\}\hat{F}_t^{(r)\prime}G^{(r)}\lambda_{ir} \equiv J_{61}^{(r)} + J_{62}^{(r)}.\end{aligned}$$

It is straightforward to show that $J_{6l}^{(r)} = O_P(h^2)O_P(\|\hat{\delta}_r\|)$ for $l = 1, 2$. Then $J_{6l}^{(r)} = O_P(h^2)O_P(\|\hat{\delta}_r\|)$. Similarly, we can show that $J_7^{(r)} = O_P(h^2)O_P(\|\hat{\delta}_r\|)$. In sum, we have established that

$$\sum_{l=1}^{7} J_l^{(r)} = \frac{1}{N^2T}\sum_{i=1}^{N}\sum_{j=1}^{N} X_i^{(r)\prime} M_{\hat{F}^{(r)}} X_j^{(r)} a_{ji}^{(r)} \hat{\delta}_r + O_P(C_{NT}^{-1} + \|\hat{\delta}_r\|^{1/2})O_P(\|\hat{\delta}_r\|).$$

The last nine terms, namely, $J_8^{(r)}$ to $J_{15}^{(r)}$, do not explicitly depend on $\hat{\delta}_r \equiv \hat{\beta}_r - \beta_r$. We will show that the term $J_9^{(r)}$ contributes to the limiting distribution of $\hat{\beta}_r - \beta_r$, the terms $J_{10}^{(r)}$ and $J_{14}^{(r)}$ contribute to the bias, and the other terms are $o_P((NTh)^{-1/2}) + O_P(C_{NT}^{-1} + \|\hat{\delta}_r\|^{1/2})O_P(\|\hat{\delta}_r\|)$. We first consider $J_8^{(r)}$. Using $M_{\hat{F}^{(r)}}F^{(r)} = M_{\hat{F}^{(r)}}[F^{(r)}H^{(r)} - \hat{F}^{(r)} + B^{(r)}]H^{(r)-1} - M_{\hat{F}^{(r)}}B^{(r)}H^{(r)-1} = -M_{\hat{F}^{(r)}}\Upsilon^{(r)}H^{(r)-1} - M_{\hat{F}^{(r)}}B^{(r)}H^{(r)-1}$, we have

$$\begin{aligned}
\omega' J_8^{(r)} &= -\frac{1}{N^2T^2}\sum_{i=1}^{N}\omega' X_i^{(r)\prime} M_{\hat{F}^{(r)}} \sum_{j=1}^{N} F^{(r)}\lambda_{jr}\varepsilon_j^{(r)\prime}\hat{F}^{(r)}G^{(r)}\lambda_{ir} \\
&= \operatorname{tr}\left(\frac{1}{NT}H^{(r)-1}\sum_{j=1}^{N}\lambda_{jr}\varepsilon_j^{(r)\prime}\hat{F}^{(r)}G^{(r)}\frac{1}{NT}\sum_{i=1}^{N}\lambda_{ir}\omega' X_i^{(r)\prime}M_{\hat{F}^{(r)}}\Upsilon^{(r)}\right) \\
&\quad + \operatorname{tr}\left(\frac{1}{NT}H^{(r)-1}\sum_{j=1}^{N}\lambda_{jr}\varepsilon_j^{(r)\prime}\hat{F}^{(r)}G^{(r)}\frac{1}{NT}\sum_{i=1}^{N}\lambda_{ir}\omega' X_i^{(r)\prime}M_{\hat{F}^{(r)}}B^{(r)}\right) \equiv J_{81}^{(r)} + J_{82}^{(r)}.
\end{aligned}$$

By Lemma A.4(i)-(iii),

$$\left\|\lambda_{ir}\omega' J_{81}^{(r)}\right\| \lesssim \frac{1}{NT}\left\|\sum_{i=1}^{N} X_i^{(r)\prime}M_{\hat{F}^{(r)}}\Upsilon^{(r)}\right\| \frac{1}{NT}\left\|\sum_{j=1}^{N}\lambda_{jr}\varepsilon_j^{(r)\prime}\hat{F}^{(r)}\right\| = \left[O_P(\|\hat{\delta}_r\|) + O_P(C_{NT}^{-2})\right]\left[O_P(\|\hat{\delta}_r\|) + O_P(C_{NT}^{-2})\right],$$

and

$$\left\|J_{82}^{(r)}\right\| \lesssim \left\|\frac{1}{NT}\sum_{i=1}^{N}\lambda_{ir}\omega' X_i^{(r)\prime}M_{\hat{F}^{(r)}}B^{(r)}\right\| \frac{1}{NT}\left\|\sum_{j=1}^{N}\lambda_{jr}\varepsilon_j^{(r)\prime}\hat{F}^{(r)}\right\| = O_P(h^2 + \|\hat{\delta}_r\|)\left[O_P(\|\hat{\delta}_r\|) + O_P(C_{NT}^{-2})\right].$$

Then $\left\|J_8^{(r)}\right\| = O_P(\|\hat{\delta}_r\| + C_{NT}^{-2} + h^2)[O_P(\|\hat{\delta}_r\|) + O_P(C_{NT}^{-2})]$. For J_9, noting that $F^{(r)\prime}\hat{F}^{(r)}G^{(r)} = (\frac{1}{N}\Lambda_r'\Lambda_r)^{-1}$, we have

$$J_9^{(r)} = -\frac{1}{N^2T^2}\sum_{i=1}^{N}X_i^{(r)\prime}M_{\hat{F}^{(r)}}\sum_{j=1}^{N}\varepsilon_j^{(r)}\lambda_{jr}'(\frac{1}{N}\Lambda_r'\Lambda_r)^{-1}\lambda_{ir} = -\frac{1}{N^2T}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}^{(r)}X_i^{(r)\prime}M_{\hat{F}^{(r)}}\varepsilon_j^{(r)}.$$

For $J_{10}^{(r)}$, we have

$$\begin{aligned}
J_{10}^{(r)} &= -\frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{j=1}^{N}X_i^{(r)\prime}M_{\hat{F}^{(r)}}\varepsilon_j^{(r)}\varepsilon_j^{(r)\prime}\hat{F}^{(r)}G^{(r)}\lambda_{ir} \\
&= -\frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{j=1}^{N}X_i^{(r)\prime}M_{\hat{F}^{(r)}}\varepsilon_j^{(r)}\varepsilon_j^{(r)\prime}\hat{F}^{(r)}(\frac{1}{T}F^{(r)\prime}\hat{F}^{(r)})^{-1}(\frac{1}{N}\Lambda_r'\Lambda_r)^{-1}\lambda_{ir}.
\end{aligned}$$

For $J_{11}^{(r)}$, we make the following decomposition:

$$\begin{aligned} J_{11}^{(r)} &= -\frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{j=1}^{N} X_i^{(r)\prime} M_{\hat{F}^{(r)}} F^{(r)} \lambda_{jr} \Delta_j^{(r)\prime} \hat{F}^{(r)} G^{(r)} \lambda_{ir} \\ &= \frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{j=1}^{N} X_i^{(r)\prime} M_{\hat{F}^{(r)}} \left[\hat{F}^{(r)} - F^{(r)}H^{(r)} - B^{(r)}\right] H^{(r)-1} \lambda_{jr} \Delta_j^{(r)\prime} \hat{F}^{(r)} G^{(r)} \lambda_{ir} \\ &+ \frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{j=1}^{N} X_i^{(r)\prime} M_{\hat{F}^{(r)}} B^{(r)} H^{(r)-1} \lambda_{jr} \Delta_j^{(r)\prime} \hat{F}^{(r)} G^{(r)} \lambda_{ir} \equiv J_{11,1}^{(r)} + J_{11,2}^{(r)}. \end{aligned}$$

By Lemma A.4(i),

$$\left|\omega' J_{11,1}^{(r)}\right| \lesssim \frac{1}{NT}\left\|\sum_{i=1}^{N} \lambda_{ir}\omega' X_i^{(r)\prime} M_{\hat{F}^{(r)}} \Upsilon^{(r)}\right\| \frac{1}{NT}\left\|\sum_{j=1}^{N} \lambda_{jr}\Delta_j^{(r)\prime}\hat{F}^{(r)}\right\| = [O_P(\|\hat{\delta}_r\|) + O_P(C_{NT}^{-2})] O_P(\|\hat{\delta}_r\| + C_{NT}^{-1}),$$

where we use the fact that

$$\begin{aligned} \left\|\frac{1}{NT}\sum_{j=1}^{N} \lambda_{jr}\Delta_j^{(r)\prime}\hat{F}^{(r)}\right\| &\le \left\|\frac{1}{NT}\sum_{j=1}^{N} \lambda_{jr}\Delta_j^{(r)\prime}\Upsilon^{(r)}\right\| + \left\|\frac{1}{NT}\sum_{j=1}^{N} \lambda_{jr}\Delta_j^{(r)\prime}(F^{(r)}H^{(r)} + B^{(r)})\right\| \\ &= O_P(h(\|\hat{\delta}_r\|^{1/2} + C_{NT}^{-1})) + O_P(h^2) = O_P(\|\hat{\delta}_r\| + h^2 + hC_{NT}^{-1}) = O_P(\|\hat{\delta}_r\| + C_{NT}^{-1}). \end{aligned}$$

For $J_{11,2}^{(r)}$, we can apply Lemmas A.2(iv) and A.3 to show that

$$\begin{aligned} J_{11,2}^{(r)} &= \frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{j=1}^{N} X_i^{(r)\prime} B^{(r)} H^{(r)-1} \lambda_{jr} \Delta_j^{(r)\prime}\left(F^{(r)}H^{(r)} + B^{(r)}\right) G^{(r)}\lambda_{ir} \\ &\quad - \frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{j=1}^{N} X_i^{(r)\prime} P_{F^{(r)}} B^{(r)} H^{(r)-1} \lambda_{jr} \Delta_j^{(r)\prime}\left(F^{(r)}H^{(r)} + B^{(r)}\right) G^{(r)}\lambda_{ir} \\ &\quad + O_P((C_{NT}^{-1} + \|\hat{\delta}_r\|^{1/2})h^3 + O_P((C_{NT}^{-2} + \|\hat{\delta}_r\|)h^2)) \\ &\equiv J_{11,21}^{(r)} + J_{11,22}^{(r)} + O_P((C_{NT}^{-1} + \|\hat{\delta}_r\|^{1/2})h^3 + O_P((C_{NT}^{-2} + \|\hat{\delta}_r\|)h^2)). \end{aligned}$$

Note that

$$\left|\omega' J_{11,21}^{(r)}\right| \lesssim \left\|\frac{1}{NT}\sum_{j=1}^{N} \lambda_{jr}\Delta_j^{(r)\prime}(F^{(r)}H^{(r)} + B^{(r)})\right\| \left\|\frac{1}{NT}\sum_{i=1}^{N} \lambda_{ir} X_i^{(r)\prime} B^{(r)}\right\| = O_P\left(h^2\right) O_P\left(h^2\right) = O_P\left(h^4\right).$$

Similarly, $\left|\omega' J_{11,21}^{(r)}\right| \lesssim O_P\left(h^4\right)$. Thus,

$$\begin{aligned} J_{11}^{(r)} &= [O_P(\|\hat{\delta}_r\|) + O_P(C_{NT}^{-2})] O_P(\|\hat{\delta}_r\| + C_{NT}^{-1}) + O_P(h^4) + O_P((C_{NT}^{-1} + \|\hat{\delta}_r\|^{1/2})h^3) + O_P((C_{NT}^{-2} + \|\hat{\delta}_r\|)h^2) \\ &= O_P(\|\hat{\delta}_r\|^2 + C_{NT}^{-3} + h^4 + C_{NT}^{-1}h^3 + \|\hat{\delta}_r\|^{1/2}h^3) = O_P(\|\hat{\delta}_r\|^2 + C_{NT}^{-3} + h^4 + C_{NT}^{-1}h^3), \end{aligned}$$

where the last equality holds because $\|\hat{\delta}_r\|^{1/2}h^3 \le (\|\hat{\delta}_r\|h^2 + h^4)/2$ by CS inequality. Analogously, we have

$$J_{12}^{(r)} = -\frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{j=1}^{N} X_i^{(r)\prime} M_{\hat{F}^{(r)}} \Delta_j^{(r)} \Delta_j^{(r)\prime} \hat{F}^{(r)} G^{(r)} \lambda_{ir}$$

$$= \frac{-1}{N^2T^2}\sum_{i=1}^{N}\sum_{j=1}^{N} X_i^{(r)\prime} M_{F^{(r)}} \Delta_j^{(r)} \Delta_j^{(r)\prime}(F^{(r)}H^{(r)} + B^{(r)})G^{(r)}\lambda_{ir} + O_P((C_{NT}^{-1} + \|\hat{\delta}_r\|^{1/2})h^3 + O_P((C_{NT}^{-2} + \|\hat{\delta}_r\|)h^2))$$
$$= O_P(h^4) + O_P((C_{NT}^{-1} + \|\hat{\delta}_r\|^{1/2})h^3 + O_P((C_{NT}^{-2} + \|\hat{\delta}_r\|)h^2))$$
$$= O_P(\|\hat{\delta}_r\|^2 + C_{NT}^{-3} + h^4 + C_{NT}^{-1}h^3 + \|\hat{\delta}_r\|^{1/2}h^3).$$

We note that the $J_{13}^{(r)}$ term is a higher order term than $J_9^{(r)}$, and $J_{15}^{(r)}$ term is a higher order term than $J_8^{(r)}$, which are both asymptotically negligible. Finally,

$$J_{14}^{(r)} = -\frac{1}{N^2T}\sum_{i=1}^{N}\sum_{j=1}^{N} X_i^{(r)\prime} M_{\hat{F}^{(r)}} \frac{1}{T}\Delta_j^{(r)} \lambda_{jr}' F^{(r)\prime}\hat{F}^{(r)} G^{(r)}\lambda_{ir} = -\frac{1}{N^2T}\sum_{i=1}^{N}\sum_{j=1}^{N} X_i^{(r)\prime} M_{\hat{F}^{(r)}} \Delta_j^{(r)} a_{ji}^{(r)}.$$

Collecting terms from $J_1^{(r)}$ to $J_{15}^{(r)}$ gives

$$\frac{1}{NT}\sum_{i=1}^{N} X_i^{(r)\prime} M_{\hat{F}^{(r)}} F^{(r)}\lambda_{ir}$$
$$= \frac{1}{N^2T}\sum_{i=1}^{N}\sum_{j=1}^{N} X_i^{(r)\prime} M_{\hat{F}^{(r)}} X_j^{(r)} a_{ji}^{(r)}\hat{\delta}_r - \frac{1}{N^2T}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^{(r)} X_i^{(r)\prime} M_{\hat{F}^{(r)}}\varepsilon_j^{(r)}$$
$$- \frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{j=1}^{N} X_i^{(r)\prime} M_{\hat{F}^{(r)}}\varepsilon_j^{(r)}\varepsilon_j^{(r)\prime}\hat{F}^{(r)}(\frac{1}{T}F^{(r)\prime}\hat{F}^{(r)})^{-1}(\frac{1}{N}\Lambda_r'\Lambda_r)^{-1}\lambda_{ir} - \frac{1}{N^2T}\sum_{i=1}^{N}\sum_{j=1}^{N} X_i^{(r)\prime} M_{\hat{F}^{(r)}}\Delta_j^{(r)} a_{ji}^{(r)}$$
$$+ O_P(C_{NT}^{-1} + \|\hat{\delta}_r\|^{1/2})O_P(\|\hat{\delta}_r\|) + O_P(C_{NT}^{-3} + h^4 + C_{NT}^{-1}h^3). \blacksquare$$

Proof of Lemma A.6. First, we consider $\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N} X_i^{(r)\prime}\left(M_{\hat{F}^{(r)}} - M_{F^{(r)}}\right)\varepsilon_i^{(r)}$. Recall that $\Upsilon^{(r)} = \hat{F}^{(r)} - F^{(r)}H^{(r)} - B^{(r)}$. Let $\Upsilon_s^{(r)} = \hat{F}_s^{(r)} - H^{(r)\prime}F_s^{(r)} - B_s^{(r)}$. Noting that $\hat{F}^{(r)} = \Upsilon^{(r)} + F^{(r)}H^{(r)} + B^{(r)}$, we make the following decomposition:

$$\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N} X_i^{(r)\prime}\left(M_{F^{(r)}} - M_{\hat{F}^{(r)}}\right)\varepsilon_i^{(r)}$$
$$= \frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N} X_i^{(r)\prime}\frac{1}{T}\left[\hat{F}^{(r)}\hat{F}^{(r)\prime} - F^{(r)}(F^{(r)\prime}F^{(r)}/T)^{-1}F^{(r)\prime}\right]\varepsilon_i^{(r)}$$
$$= \frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N} X_i^{(r)\prime}\frac{1}{T}\Upsilon^{(r)}\Upsilon^{(r)\prime}\varepsilon_i^{(r)} + \frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N} X_i^{(r)\prime}\frac{1}{T}\Upsilon^{(r)}H^{(r)\prime}F^{(r)\prime}\varepsilon_i^{(r)} + \frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N} X_i^{(r)\prime}\frac{1}{T}F^{(r)}H^{(r)}\Upsilon^{(r)\prime}\varepsilon_i^{(r)}$$
$$+ \frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N} X_i^{(r)\prime}\frac{1}{T}F^{(r)}\left[H^{(r)}H^{(r)\prime} - (F^{(r)\prime}F^{(r)}/T)^{-1}\right]F^{(r)\prime}\varepsilon_i^{(r)} + \frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N} X_i^{(r)\prime}\frac{1}{T}\Upsilon^{(r)}B^{(r)\prime}\varepsilon_i^{(r)}$$
$$+ \frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N} X_i^{(r)\prime}\frac{1}{T}B^{(r)}\Upsilon^{(r)}\varepsilon_i^{(r)} + \frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N} X_i^{(r)\prime}\frac{1}{T}F^{(r)}H^{(r)}B^{(r)\prime}\varepsilon_i^{(r)}$$
$$+ \frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N} X_i^{(r)\prime}\frac{1}{T}B^{(r)}H^{(r)\prime}F^{(r)\prime}\varepsilon_i^{(r)} + \frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N} X_i^{(r)\prime}\frac{1}{T}B^{(r)}B^{(r)\prime}\varepsilon_i^{(r)} \equiv \sum_{l=1}^{9} A_l^{(r)}.$$

We consider these nine terms one by one. First, for $A_1^{(r)}$, we have

$$\left\|A_1^{(r)}\right\| \quad \leq \quad \sqrt{NTh}\left\|\frac{1}{N}\sum_{i=1}^{N}\frac{1}{T}X_i^{(r)\prime}\Upsilon^{(r)}\frac{1}{T}\Upsilon^{(r)\prime}\varepsilon_i^{(r)}\right\|$$

$$
\begin{aligned}
&\leq \sqrt{NTh}\left\{\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}X_i^{(r)\prime}\Upsilon^{(r)}\right\|^2\right\}^{1/2}\left\{\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\Upsilon^{(r)\prime}\varepsilon_i^{(r)}\right\|^2\right\}^{1/2} \\
&\leq \sqrt{NTh}\left\{\frac{1}{NT}\sum_{i=1}^{N}\left\|X_i^{(r)}\right\|^2\right\}^{1/2}\frac{1}{T^{1/2}}\left\|\Upsilon^{(r)}\right\|\left\{\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\Upsilon^{(r)\prime}\varepsilon_i^{(r)}\right\|^2\right\}^{1/2} \\
&= \sqrt{NTh}O_P(C_{NT}^{-1}+\|\hat{\delta}_r\|^{1/2})O_P(C_{NT}^{-2}+\|\hat{\delta}_r\|)=\sqrt{NTh}O_P(C_{NT}^{-3}+\|\hat{\delta}_r\|^{3/2}),
\end{aligned}
$$

where the first equality follows from Lemmas A.3(i) and A.4(iii). For $A_2^{(r)}$, we have

$$
\begin{aligned}
\left|\omega' A_2^{(r)}\right| &= \sqrt{NTh}\left|\frac{1}{NT}\sum_{i=1}^{N}\omega' X_i^{(r)\prime}\frac{1}{T}\Upsilon^{(r)}H^{(r)\prime}F^{(r)\prime}\varepsilon_i^{(r)}\right| \\
&= \sqrt{NTh}\left|\frac{1}{T}\sum_{s=1}^{T}\Upsilon_s^{(r)\prime}H^{(r)\prime}\left(\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}F_t^{(r)}\varepsilon_{it}^{(r)}\omega' X_{is}^{(r)}\right)\right| \\
&\lesssim \sqrt{NTh}\left[\frac{1}{T}\sum_{s=1}^{T}\left\|\Upsilon_s^{(r)}\right\|^2\right]^{1/2}\left[\frac{1}{T}\sum_{s=1}^{T}k_{h,sr}^{*}\left\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}k_{h,tr}^{*}F_t\varepsilon_{it}\omega' X_{is}\right\|^2\right]^{1/2} \\
&= \sqrt{NTh}O_P(C_{NT}^{-1}+\|\hat{\delta}_r\|^{1/2})O_P((NTh)^{-1/2}+(Th)^{-1}),
\end{aligned}
$$

where we use Lemma A.3(i) and the fact that

$$
\begin{aligned}
&\frac{1}{T}\sum_{s=1}^{T}k_{h,sr}^{*}\left\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}k_{h,tr}^{*}F_t\varepsilon_{it}X_{is}'\omega\right\|^2 \\
\leq\ &\frac{2}{T}\sum_{s=1}^{T}k_{h,sr}^{*}\left\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}k_{h,tr}^{*}[F_t\varepsilon_{it}X_{is}'-E(F_t\varepsilon_{it}X_{is}')]\right\|^2+\frac{2}{T}\sum_{s=1}^{T}k_{h,sr}^{*}\left\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}k_{h,tr}^{*}E\left(F_t\varepsilon_{it}X_{is}'\right)\right\|^2 \\
=\ &O_P\left((NTh)^{-1}\right)+O_P((Th)^{-2})
\end{aligned}
$$

by noticing that $\max_s\left\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}k_{h,tr}^{*}E\left(F_t\varepsilon_{it}X_{is}'\right)\right\|^2\leq\max_t k_{h,tr}^{*2}\left|\max_s\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\|E\left(F_t\varepsilon_{it}X_{is}'\right)\|\right|^2$ $\leq h^{-2}O_P(T^{-2})$.

For $A_3^{(r)}$, we have

$$
\begin{aligned}
A_3^{(r)} &= \frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}X_i^{(r)\prime}\frac{1}{T}F^{(r)}H^{(r)}\left[\hat{F}^{(r)}-F^{(r)}H^{(r)}-B^{(r)}\right]'\varepsilon_i^{(r)} \\
&= \frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\frac{X_i^{(r)\prime}F^{(r)}}{T}[\frac{1}{T}F^{(r)\prime}F^{(r)}]^{-1}\left[\hat{F}^{(r)}H^{(r)-1}-F^{(r)}-B^{(r)}H^{(r)-1}\right]'\varepsilon_i^{(r)} \\
&+\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\frac{X_i^{(r)\prime}F^{(r)}}{T}\left\{H^{(r)}H^{(r)\prime}-[\frac{1}{T}F^{(r)\prime}F^{(r)}]^{-1}\right\}\left[\hat{F}^{(r)}H^{(r)-1}-F^{(r)}-B^{(r)}H^{(r)-1}\right]'\varepsilon_i^{(r)}\equiv\sum_{\ell=1}^{2}A_{3\ell}^{(r)}.
\end{aligned}
$$

By Lemma A.4(iv), $A_{31}^{(r)}=\sqrt{\frac{Th}{N}}\psi_{NT}^{(r)}+O_P(\|\hat{\delta}_r\|)+\sqrt{Th}O_P(C_{NT}^{-2})$, where

$$
\psi_{NT}^{(r)}=\frac{1}{N}\sum_{i=1}^{N}\sum_{k=1}^{N}\frac{X_i^{(r)\prime}F^{(r)}}{T}\left(\frac{F^{(r)\prime}F^{(r)}}{T}\right)^{-1}\left(\frac{\Lambda_r'\Lambda_r}{N}\right)^{-1}\lambda_{kr}\left(\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}^{(r)}\varepsilon_{kt}^{(r)}\right).
$$

Let $\Xi^{(r)} = H^{(r)}H^{(r)\prime} - [\frac{1}{T}F^{(r)\prime}F^{(r)}]^{-1}$. Note that

$$\begin{aligned}\frac{1}{T}F^{(r)\prime}F^{(r)} - \Sigma_F &= \frac{1}{T}\sum_{t=1}^{T}k_{h,tr}^{*}F_tF_t' - \Sigma_F = \frac{1}{T}\sum_{t=1}^{T}k_{h,tr}^{*}[F_tF_t' - E(F_tF_t')] + (\frac{1}{T}\sum_{t=1}^{T}k_{h,tr}^{*} - 1)\Sigma_F \\ &= O_P((Th)^{-1/2}) + O((Th)^{-1}) = O_P((Th)^{-1/2})\end{aligned}$$

by our assumptions and the approximation property of Riemann summation to a definite integral. This fact, along with Lemma A.2(iv), implies that

$$\left\|\Xi^{(r)}\right\| = O_P(C_{NT}^{-1} + \|\hat{\delta}_r\|). \tag{C.6}$$

Then we have

$$\begin{aligned}\left\|A_{32}^{(r)}\right\| &\lesssim \sqrt{NTh}\left\|\Xi^{(r)}\right\|\left\{\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{X_i^{(r)\prime}F^{(r)}}{T}\right\|^2\right\}^{1/2}\left\{\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\Upsilon^{(r)\prime}\varepsilon_i^{(r)}\right\|^2\right\}^{1/2} \\ &= \sqrt{NTh}O_P(C_{NT}^{-1} + \|\hat{\delta}_r\|)O_P(1)O_P(C_{NT}^{-2} + \|\hat{\delta}_r\|) = \sqrt{NTh}O_P(C_{NT}^{-3} + \|\hat{\delta}_r\|^2 + C_{NT}^{-1}\|\hat{\delta}_r\|)\end{aligned}$$

by the CS inequality and A.4(iv). Thus $A_3^{(r)} = \sqrt{\frac{Th}{N}}\psi_{NT}^{(r)} + O_P(\|\hat{\delta}_r\|) + \sqrt{Th}O_P(C_{NT}^{-2}) + \sqrt{NTh}O_P(C_{NT}^{-3} + \|\hat{\delta}_r\|^2 + C_{NT}^{-1}\|\hat{\delta}_r\|)$.

For $A_4^{(r)}$, we have

$$\begin{aligned}\left|\omega' A_4^{(r)}\right| &= \mathrm{tr}\left[\Xi^{(r)}\left(\frac{\sqrt{NTh}}{NT^2}\sum_{i=1}^{N}F^{(r)\prime}\varepsilon_i^{(r)}\omega' X_i^{(r)\prime}F^{(r)}\right)\right] \leq \left\|\Xi^{(r)}\right\|\frac{\sqrt{NTh}}{NT^2}\left\|\sum_{i=1}^{N}F^{(r)\prime}\varepsilon_i^{(r)}\omega' X_i^{(r)\prime}F^{(r)}\right\| \\ &= O_P(C_{NT}^{-1} + \|\hat{\delta}_r\|)O_P(1 + (Th/N)^{-1/2}).\end{aligned}$$

For $A_5^{(r)}$, we have

$$\begin{aligned}\left\|A_5^{(r)}\right\| &= \frac{\sqrt{NTh}}{NT^2}\left\|\sum_{i=1}^{N}X_i^{(r)\prime}\Upsilon^{(r)}B^{(r)\prime}\varepsilon_i^{(r)}\right\| \leq \sqrt{NTh}\left\{\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}X_i^{(r)\prime}\Upsilon^{(r)}\right\|^2\right\}^{1/2}\left\{\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}B^{(r)\prime}\varepsilon_i^{(r)}\right\|^2\right\}^{1/2} \\ &= \sqrt{NTh}O_P(C_{NT}^{-2} + \|\hat{\delta}_r\|)O_P((T/h)^{-1/2}).\end{aligned}$$

Next,

$$\begin{aligned}\left\|A_6^{(r)}\right\| &= \frac{\sqrt{NTh}}{NT^2}\left\|\sum_{i=1}^{N}X_i^{(r)\prime}B^{(r)}\Upsilon^{(r)\prime}\varepsilon_i^{(r)}\right\| \leq \sqrt{NTh}\left\{\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}X_i^{(r)\prime}B^{(r)}\right\|^2\right\}^{1/2}\left\{\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\Upsilon^{(r)\prime}\varepsilon_i^{(r)}\right\|^2\right\}^{1/2} \\ &= \sqrt{NTh}O_P(h)O_P(C_{NT}^{-2} + \|\hat{\delta}_r\|).\end{aligned}$$

For $A_7^{(r)}$, we have

$$\begin{aligned}\left|\omega' A_7^{(r)}\right| &= \frac{\sqrt{NTh}}{NT^2}\left|\mathrm{tr}\left(H^{(r)}\sum_{i=1}^{N}B^{(r)\prime}\varepsilon_i^{(r)}\omega' X_i^{(r)\prime}F^{(r)}\right)\right| \lesssim \frac{\sqrt{NTh}}{NT^2}\left\|\sum_{i=1}^{N}B^{(r)\prime}\varepsilon_i^{(r)}\omega' X_i^{(r)\prime}F^{(r)}\right\| \\ &= \frac{\sqrt{NTh}}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}B_t^{(r)}\varepsilon_{it}^{(r)}F_s^{(r)\prime}X_{is}^{(r)\prime}\omega = O_P(N^{1/2}T^{-1/2}h) = o_P(1),\end{aligned}$$

where we use the fact that $B_t^{(r)} = k_{h,tr}^{*1/2}\left[C_{1t}^{(r)}\frac{t-r}{T} + C_{2t}^{(r)}h^2 + C_{3t}^{(r)}(\frac{t-r}{T})^2\right]$. Similarly, $A_8^{(r)} = O_P(N^{1/2}T^{-1/2}h) = o_P(1)$. Finally, for $A_9^{(r)}$, we can readily show that

$$\begin{aligned} A_9^{(r)} &= \frac{\sqrt{NTh}}{NT^2}\sum_{i=1}^N X_i^{(r)\prime}B^{(r)}B^{(r)\prime}\varepsilon_i^{(r)} = \frac{\sqrt{NTh}}{NT^2}\sum_{i=1}^N\sum_{t=1}^T\sum_{s=1}^T X_{ir}^{(r)}B_t^{(r)\prime}B_s^{(r)}\varepsilon_{is}^{(r)} \\ &= \frac{\sqrt{NTh}}{NT^2}\sum_{i=1}^N\sum_{t=1}^T\sum_{s=1}^T k_{h,tr}^* k_{h,sr}^* X_{it}C_{1t}^{(r)\prime}C_{1s}^{(r)}\varepsilon_{is}\frac{t-r}{T}\frac{s-r}{T} + o_P(1) \equiv A_{91}^{(r)} + o_P(1). \end{aligned}$$

Recall that $C_{1t}^{(r)} = \hat{V}_{NT}^{(r)-1}H^{(r)\prime}\Sigma_F(\frac{1}{N}\Lambda_r^{\prime}A_{1,tr})$ and $A_{1,tr} = X_t\beta_r^{(1)} + \Lambda_r^{(1)}F_t$. Let $\bar{C}_{1t}^{(r)} = \hat{V}_{NT}^{(r)-1}H^{(r)\prime}\Sigma_F E(\frac{1}{N}\Lambda_r^{\prime}A_{1,tr})$. Then we can show that $A_{91}^{(r)} = \bar{A}_{91}^{(r)} + o_P(1)$, where

$$\bar{A}_{91}^{(r)} = \frac{\sqrt{NTh}}{NT^2}\sum_{i=1}^N\sum_{t=1}^T\sum_{s=1}^T k_{h,tr}^* k_{h,sr}^* E(X_{it}\bar{C}_{1t}^{(r)\prime})\bar{C}_{1s}^{(r)}\varepsilon_{is}\frac{t-r}{T}\frac{s-r}{T} = \frac{\sqrt{NTh}}{NT}\sum_{i=1}^N\sum_{s=1}^T k_{h,sr}^* A_{iT}^{(r)}\bar{C}_{1s}^{(r)}\varepsilon_{is}\frac{s-r}{T}$$

and $A_{iT}^{(r)} = \frac{1}{T}\sum_{t=1}^T k_{h,tr}^* E(X_{it}\bar{C}_{1t}^{(r)\prime})\frac{t-r}{T} = O\left(h\right)$. By straightforward moment calculations, we have $E(\bar{A}_{91}^{(r)}) = 0$ and $E\left\|\bar{A}_{91}^{(r)}\right\|^2 = O\left(h^4\right)$. Then $\bar{A}_{91}^{(r)} = O_P\left(h^2\right)$ and $A_9^{(r)} = o_P(1)$.

In sum, we have

$$\begin{aligned} &\frac{\sqrt{NTh}}{NT}\sum_{i=1}^N X_i^{(r)\prime}\left(M_{F^{(r)}} - M_{\hat{F}^{(r)}}\right)\varepsilon_i^{(r)} \\ &= \sqrt{\frac{Th}{N}}\psi_{NT}^{(r)} + \sqrt{NTh}O_P((C_{NT}^{-1}+h)\|\hat{\delta}_r\| + \|\hat{\delta}_r\|^{3/2}) + \sqrt{NTh}(Th)^{-1}O_P(\|\hat{\delta}_r\|^{1/2}) + o_P(1) \end{aligned}$$

where we use the fact that $(NTh)^{1/2}C_{NT}^{-2}(C_{NT}^{-1}+h) = o(1)$, $(\sqrt{Th}+\sqrt{N}h)C_{NT}^{-2} = o(1)$, $(Th/N)^{-1/2}C_{NT}^{-1} = o(1)$ and $Nh^2T^{-1} = o(1)$.

Recall that $V_i^{(r)} = \frac{1}{N}\sum_{k=1}^N a_{ik}^{(r)}X_k^{(r)}$. Then replacing $X_i^{(r)}$ with $V_i^{(r)}$, the same argument leads to

$$\begin{aligned} &\frac{\sqrt{NTh}}{NT}\sum_{i=1}^N V_i^{(r)\prime}\left(M_{F^{(r)}} - M_{\hat{F}^{(r)}}\right)\varepsilon_i^{(r)} \\ =\;& \sqrt{\frac{Th}{N}}\psi_{NT}^{*(r)} + \sqrt{NTh}O_P((C_{NT}^{-1}+h)\|\hat{\delta}_r\| + \|\hat{\delta}_r\|^{3/2}) + \sqrt{NTh}(Th)^{-1}O_P(\|\hat{\delta}_r\|^{1/2}) + o_P(1), \end{aligned}$$

where

$$\psi_{NT}^{*(r)} = \frac{1}{N}\sum_{i=1}^N\sum_{k=1}^N \frac{V_i^{(r)\prime}F^{(r)}}{T}\left(\frac{F^{(r)\prime}F^{(r)}}{T}\right)^{-1}\left(\frac{\Lambda_r^{\prime}\Lambda_r}{N}\right)^{-1}\lambda_{kr}\left(\frac{1}{T}\sum_{t=1}^T\varepsilon_{it}^{(r)}\varepsilon_{kt}^{(r)}\right).$$

Define $\xi_{NT}^{(r)} = -[\psi_{NT}^{(r)} - \psi_{NT}^{*(r)}]$. Then, we have

$$\begin{aligned} &\frac{\sqrt{NTh}}{NT}\sum_{i=1}^N\left[X_i^{(r)\prime}M_{\hat{F}^{(r)}} - \frac{1}{N}\sum_{k=1}^N a_{ik}^{(r)}X_k^{(r)\prime}M_{\hat{F}^{(r)}}\right]\varepsilon_i^{(r)} \\ &= \frac{\sqrt{NTh}}{NT}\sum_{i=1}^N\left[X_i^{(r)\prime}M_{F^{(r)}} - \frac{1}{N}\sum_{k=1}^N a_{ik}^{(r)}X_k^{(r)\prime}M_{F^{(r)}}\right]\varepsilon_i^{(r)} \\ &+ \sqrt{\frac{Th}{N}}\xi_{NT}^{(r)} + \sqrt{NTh}O_P((C_{NT}^{-1}+h)\|\hat{\delta}_r\| + \|\hat{\delta}_r\|^{3/2}) + \sqrt{NTh}(Th)^{-1}O_P(\|\hat{\delta}_r\|^{1/2}) + o_P(1). \end{aligned}$$

This completes the proof of the lemma. ■

Proof of Lemma A.7. We consider the term $\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}X_i^{(r)\prime}M_{\hat{F}^{(r)}}\Delta_i^{(r)}$ first.

$$\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}X_i^{(r)\prime}M_{\hat{F}^{(r)}}\Delta_i^{(r)}=\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}X_i^{(r)\prime}\Delta_i^{(r)}-\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}X_i^{(r)\prime}\frac{\hat{F}^{(r)}\hat{F}^{(r)\prime}}{T}\Delta_i^{(r)}.$$

We study the two terms on the right hand side of the above equation in turn. Recall that $A_{l,itr}$ denote the ith element of $A_{l,tr}=X_t\beta_r^{(l)}+\Lambda_r^{(l)}F_t$ for $l=1,2$. Noting that

$$\Delta_{i,t}^{(r)}=k_{h,tr}^{*1/2}\Delta_i\left(t,r\right)=k_{h,tr}^{*1/2}\left[X_{it}^{\prime}d_0\left(t,r\right)+F_t^{\prime}d_i\left(t,r\right)\right]=k_{h,tr}^{*1/2}\{A_{1,itr}\frac{t-r}{T}+A_{2,itr}(\frac{t-r}{T})^2+O_P(h^3)\},$$

for the term $\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}X_i^{(r)\prime}\Delta_i^{(r)}$, we have

$$\begin{aligned}
&\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}X_i^{(r)\prime}\Delta_i^{(r)}=\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}X_{it}^{(r)}\Delta_{i,t}^{(r)}\\
&=\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}k_{h,tr}^{*}X_{it}[A_{1,itr}\frac{t-r}{T}+A_{2,itr}(\frac{t-r}{T})^2]+o_P(1)\\
&=\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}k_{h,tr}^{*}\left[X_{it}A_{1,itr}-E(X_{it}A_{1,itr})\right]\frac{t-r}{T}+\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}k_{h,tr}^{*}\left[X_{it}A_{2,itr}-E(X_{it}A_{2,itr})\right](\frac{t-r}{T})^2\\
&+\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}k_{h,tr}^{*}E(X_{it}A_{1,itr})\frac{t-r}{T}+\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}k_{h,tr}^{*}E(X_{it}A_{2,itr})(\frac{t-r}{T})^2+o_P(1)\equiv\sum_{\ell=1}^{4}S_\ell+o_P(1).
\end{aligned}$$

By standard variance calculation and Chebyshev inequality

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}k_{h,tr}^{*}\left[X_{it}A_{1,itr}-E(X_{it}A_{1,itr})\right](\frac{t-r}{T})^l=O_P((NTh)^{-1/2}h^l).$$

This implies that $S_l=O_P(h^l)$ for $l=1,2$. By the property of Riemann integral,

$$\begin{aligned}
S_3&=\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}k_{h,tr}^{*}E(X_{it}A_{1,itr})\frac{t-r}{T}\\
&=\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}k_{h,tr}^{*}\left[E(X_{it}X_{it}^{\prime})\beta_r^{(1)}+E(X_{it}F_t^{\prime})\lambda_{ir}^{(1)}\right]\frac{t-r}{T}\\
&=\frac{\sqrt{NTh}}{N}\sum_{i=1}^{N}\left[E(X_{i1}X_{i1}^{\prime})\beta_r^{(1)}+E(X_{i1}F_t^{\prime})\lambda_{i1}^{(1)}\right]\left\{\int uk\left(u\right)du+\frac{1}{T}\right\}=O(\sqrt{Nh/T})=o\left(1\right).
\end{aligned}$$

It follows that $\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}X_i^{(r)\prime}\Delta_i^{(r)}=\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}k_{h,tr}^{*}E(X_{it}A_{2,itr})\left(\frac{t-r}{T}\right)^2+o_P(1)$.

Next, we study the term $\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}X_i^{(r)\prime}\frac{\hat{F}^{(r)}\hat{F}^{(r)\prime}}{T}\Delta_i^{(r)}$. Noting that $\hat{F}_t^{(r)}=\Upsilon_t^{(r)}+H^{(r)\prime}F_t^{(r)}+B_t^{(r)}$, we make the following decomposition:

$$\begin{aligned}
&\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}X_i^{(r)\prime}\frac{\hat{F}^{(r)}\hat{F}^{(r)\prime}}{T}\Delta_i^{(r)}=\frac{\sqrt{NTh}}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}X_{it}^{(r)}\hat{F}_t^{(r)\prime}\hat{F}_s^{(r)}\Delta_{i,s}^{(r)}\\
&=\frac{\sqrt{NTh}}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}X_{it}^{(r)}\{\Upsilon_t^{(r)\prime}\Upsilon_s^{(r)}+F_t^{(r)\prime}H^{(r)}\Upsilon_s^{(r)}+B_t^{(r)\prime}\Upsilon_s^{(r)}+\Upsilon_t^{(r)\prime}H^{(r)\prime}F_s^{(r)}+\Upsilon_t^{(r)\prime}B_s^{(r)}
\end{aligned}$$

$$+ F_t^{(r)\prime} H^{(r)} H^{(r)\prime} F_s^{(r)} + B_t^{(r)\prime} H^{(r)\prime} F_s^{(r)} + F_t^{(r)\prime} H^{(r)} B_s^{(r)} + B_t^{(r)\prime} B_s^{(r)} \} \Delta_{i,s}^{(r)} \equiv \sum_{l=1}^{9} D_l^{(r)},$$

where $D_1^{(r)} = \frac{\sqrt{NTh}}{NT^2} \sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T X_{it}^{(r)} \Upsilon_t^{(r)\prime} \Upsilon_s^{(r)} \Delta_{i,s}^{(r)}$ and other terms are analogously defined. For $D_1^{(r)}$, we have

$$\begin{aligned} \left\| D_1^{(r)} \right\| &\leq \sqrt{NTh} \left\{ \frac{1}{N} \sum_{i=1}^N \left\| \frac{1}{T} \sum_{t=1}^T X_{it}^{(r)} \Upsilon_t^{(r)\prime} \right\|^2 \right\}^{1/2} \left\{ \frac{1}{N} \sum_{i=1}^N \left\| \frac{1}{T} \sum_{s=1}^T \Upsilon_s^{(r)} \Delta_i(s,r) \right\|^2 \right\}^{1/2} \\ &= \sqrt{NTh} O_P((C_{NT}^{-2} + \|\hat{\delta}_r\|)^2 h) \end{aligned}$$

by similar arguments as used to obtain Lemma A.3(iv) and (v). Following the proof of Lemma A.4(iv) and (v), we can also show that

$$D_l^{(r)} = \sqrt{NTh} O_P((C_{NT}^{-2} + \|\hat{\delta}_r\|) h^2) \text{ for } l = 2,\ 3,\ 4,\ 5 \text{ and } D_9^{(r)} = \sqrt{NTh} O_P(h^4 + (Th)^{-1/2} h^3) = o_P(1).$$

For $D_6^{(r)}$, we have

$$\begin{aligned} D_6^{(r)} &= \frac{\sqrt{NTh}}{NT^2} \sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T X_{it}^{(r)} F_t^{(r)\prime} H^{(r)} H^{(r)\prime} F_s^{(r)} \Delta_{i,s}^{(r)} \\ &= \sum_{l=1}^2 \frac{\sqrt{NTh}}{NT^2} \sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T k_{h,tr}^* k_{h,sr}^* X_{it} F_t' H^{(r)} H^{(r)\prime} F_s A_{1,isr} (\frac{s-r}{T})^l + o_p(1) \equiv \sum_{l=1}^2 D_{6l}^{(r)} + o_p(1). \end{aligned}$$

For $D_{61}^{(r)}$, we have

$$\begin{aligned} \omega' D_{61}^{(r)} &= \frac{\sqrt{NTh}}{NT^2} \left\| \operatorname{tr} \left\{ H^{(r)} H^{(r)\prime} \sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T F_s A_{1,isr} \frac{s-r}{T} k_{h,tr}^* k_{h,sr}^* \omega' X_{it} F_t' \right\} \right\| \\ &\lesssim \frac{\sqrt{NTh}}{NT^2} \left\| \sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T k_{h,sr}^* F_s A_{1,isr} \frac{s-r}{T} k_{h,tr}^* \omega' X_{it} F_t' \right\| \\ &\leq \frac{\sqrt{NTh}}{NT^2} \left\| \sum_{i=1}^N \sum_{t=1}^T \left\{ \sum_{s=1}^T k_{h,sr}^* E(F_s A_{1,isr}) \frac{s-r}{T} \right\} k_{h,tr}^* \omega' E(X_{it} F_t') \right\| + \sqrt{NTh} O_P\left((Th)^{-1} h + T^{-1} \right) \\ &= \sqrt{NTh} O_P(T^{-1}) + O_P(\sqrt{Nh/T}) = o_P(1). \end{aligned}$$

It follows that

$$D_6^{(r)} = \frac{\sqrt{NTh}}{NT^2} \sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T k_{h,tr}^* k_{h,sr}^* X_{it} F_t' H^{(r)} H^{(r)\prime} F_s A_{2,isr} (\frac{s-r}{T})^2 + o_p(1).$$

Similarly, we can show that $D_7^{(r)} = o_p(1)$. For $D_8^{(r)}$, we have

$$\begin{aligned} D_8^{(r)} &= \frac{\sqrt{NTh}}{NT^2} \sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T X_{it}^{(r)} F_t^{(r)\prime} H^{(r)} B_s^{(r)} \Delta_{i,s}^{(r)} \\ &= \frac{\sqrt{NTh}}{NT^2} \sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T k_{h,tr}^* k_{h,sr}^* X_{it} F_t' H^{(r)} C_{1s}^{(r)} A_{1,isr} (\frac{s-r}{T})^2 + o_p(1). \end{aligned}$$

It follows that

$$\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}X_i^{(r)\prime}M_{\hat F^{(r)}}\Delta_i^{(r)}$$
$$=\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}k_{h,tr}^{*}E(X_{it}A_{2,itr})(\frac{t-r}{T})^2$$
$$+\frac{\sqrt{NTh}}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,tr}^{*}k_{h,sr}^{*}X_{it}F_t'H^{(r)}[H^{(r)\prime}F_sA_{2,isr}+C_{1s}^{(r)}A_{1,isr}](\frac{s-r}{T})^2+o_p(1).$$

Recall that $V_i^{(r)}=N^{-1}\sum_{k=1}^{N}a_{ik}^{(r)}X_k^{(r)}$ and $V_{it}^{(r)}=N^{-1}\sum_{k=1}^{N}a_{ik}^{(r)}X_{kt}^{(r)}\equiv k_{h,tr}^{*1/2}V_{it,r}$ where $V_{it,r}=N^{-1}\sum_{k=1}^{N}a_{ik}^{(r)}X_{kt}$. By replacing $X_i^{(r)}$ with $V_i^{(r)}$ in the above analyses, we can show that

$$\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\frac{1}{N}\sum_{k=1}^{N}a_{ik}^{(r)}X_k^{(r)\prime}M_{\hat F^{(r)}}\Delta_i^{(r)}=\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}V_i^{(r)}M_{\hat F^{(r)}}\Delta_i^{(r)}$$
$$=\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}k_{h,tr}^{*}E(V_{it,r}A_{2,itr})(\frac{t-r}{T})^2$$
$$+\frac{\sqrt{NTh}}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,tr}^{*}k_{h,sr}^{*}V_{it,r}F_t'H^{(r)}[H^{(r)\prime}F_sA_{2,isr}+C_{1s}^{(r)}A_{1,isr}](\frac{s-r}{T})^2+o_p\left(1\right).$$

In sum, we have

$$\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\left[X_i^{(r)\prime}M_{\hat F^{(r)}}-\frac{1}{N}\sum_{k=1}^{N}a_{ik}^{(r)}X_k^{(r)\prime}M_{\hat F^{(r)}}\right]\Delta_i^{(r)}=\sqrt{NTh}B_{1\beta}^{(r)}+o_p(1),$$

with

$$B_{1\beta}^{(r)}=\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}k_{h,tr}E\left[(X_{it}-V_{it,r})A_{2,itr}\right](\frac{t-r}{T})^2$$
$$+\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,tr}k_{h,sr}(X_{it}-V_{it,r})F_t'H^{(r)}[H^{(r)\prime}F_sA_{2,isr}+C_{1s}^{(r)}A_{1,isr}](\frac{s-r}{T})^2.\ \blacksquare$$

Proof of Lemma A.8. We note that $M_{F^{(r)}}(X_i^{(r)}-\frac{1}{N}\sum_{j=1}^{N}X_j^{(r)}a_{ij}^{(r)})\equiv Z_i^{(r)}$ is a $T\times P$ matrix. Stack $Z_i^{(r)}$ into a $T\times N\times P$ three dimensional matrix $\mathbf{Z}^{(r)}$. Following Moon and Weidner (2017), we denote the pth sheet of $\mathbf{Z}^{(r)}$ as $\mathbf{Z}_k^{(r)}$, where $p=1,...,P$. Recall that $\mathbf{X}_p^{(r)}$ denotes the pth sheet of the three dimensional regressor matrix $\mathbf{X}^{(r)}$, $\bar{\mathbf{X}}_p^{(r)}=E(\mathbf{X}_p^{(r)})$, $\tilde{\mathbf{X}}_p^{(r)}=\mathbf{X}_p^{(r)}-\bar{\mathbf{X}}_p^{(r)}$, and $\mathfrak{X}_p^{(r)}=M_{F^{(r)}}\bar{\mathbf{X}}_p^{(r)}M_{\Lambda_r}+\tilde{\mathbf{X}}_p^{(r)}$. It is easy to verify that

$$\mathbf{Z}_p^{(r)}=M_{F^{(r)}}\mathbf{X}_p^{(r)}M_{\Lambda_r}=\mathfrak{X}_p^{(r)}-P_{F^{(r)}}\tilde{\mathbf{X}}_p^{(r)}-\tilde{\mathbf{X}}_p^{(r)}P_{\Lambda_r}+P_{F^{(r)}}\tilde{\mathbf{X}}_p^{(r)}P_{\Lambda_r}.$$

Then the pth entry of $\frac{\sqrt{NTh}}{NT}\sum_{i=1}^{N}\left[X_i^{(r)\prime}-\frac{1}{N}\sum_{j=1}^{N}a_{ij}^{(r)}X_j^{(r)\prime}\right]M_{F^{(r)}}\varepsilon_i^{(r)}$ can be written as

$$\frac{\sqrt{NTh}}{NT}\mathrm{tr}\left(M_{F^{(r)}}\mathbf{X}_p^{(r)}M_{\Lambda_r}\varepsilon^{(r)\prime}\right)$$
$$=\frac{\sqrt{NTh}}{NT}\mathrm{tr}\left([\mathfrak{X}_p^{(r)}-P_{F^{(r)}}\tilde{\mathbf{X}}_p^{(r)}-\tilde{\mathbf{X}}_p^{(r)}P_{\Lambda_r}+P_{F^{(r)}}\tilde{\mathbf{X}}_p^{(r)}P_{\Lambda_r}]\varepsilon^{(r)\prime}\right)$$
$$=\frac{\sqrt{NTh}}{NT}\mathrm{tr}\left(\mathfrak{X}_p^{(r)}\varepsilon^{(r)\prime}\right)-\frac{\sqrt{NTh}}{NT}\mathrm{tr}\left(P_{F^{(r)}}E[\tilde{\mathbf{X}}_p^{(r)}\varepsilon^{(r)\prime}]\right)-\frac{\sqrt{NTh}}{NT}\mathrm{tr}\left(P_{F^{(r)}}\left(\tilde{\mathbf{X}}_p^{(r)}\varepsilon^{(r)\prime}-E[\tilde{\mathbf{X}}_p^{(r)}\varepsilon^{(r)\prime}]\right)\right)$$

$$-\frac{\sqrt{NTh}}{NT}\text{tr}\left(\tilde{\mathbf{X}}_k^{(r)}P_{\Lambda_r}\varepsilon^{(r)\prime}\right)+\frac{\sqrt{NTh}}{NT}\text{tr}\left(P_{F^{(r)}}\tilde{\mathbf{X}}_k^{(r)}P_{\Lambda_r}\varepsilon^{(r)\prime}\right)\equiv\sum_{\ell=1}^{5}I_\ell.$$

Note that $I_1=\frac{\sqrt{NTh}}{NT}\text{tr}(\mathfrak{X}_k^{(r)}\varepsilon^{(r)\prime})=\frac{\sqrt{NTh}}{NT}\sum_{i=1}^N\sum_{t=1}^T\mathfrak{X}_{k,it}^{(r)}\varepsilon_{it}^{(r)}$ and $I_2=\frac{\sqrt{NTh}}{NT}\text{tr}\Big(P_{F^{(r)}}E[\mathbf{X}_k^{(r)}\varepsilon^{(r)\prime}]\Big)=\sqrt{\frac{N}{Th}}\tilde{B}_{2\beta,k}^{(r)}$, where $\tilde{B}_{2\beta,k}^{(r)}=-\frac{h}{N}\text{tr}\Big(P_{F^{(r)}}E[\mathbf{X}_k^{(r)}\varepsilon^{(r)\prime}]\Big)$. By Lemma A.10 below, $I_l=o_P(1)$ for $l=3,4,5$. It follows that

$$\begin{aligned}\frac{\sqrt{NTh}}{NT}\sum_{i=1}^N\left[X_i^{(r)\prime}-\frac{1}{N}\sum_{j=1}^N a_{ij}^{(r)}X_j^{(r)\prime}\right]M_{F^{(r)}}\varepsilon_i^{(r)}-\sqrt{\frac{N}{Th}}\tilde{B}_{2\beta}^{(r)}&=\frac{\sqrt{NTh}}{NT}\sum_{i=1}^N\sum_{t=1}^T\mathfrak{X}_{it}^{(r)}\varepsilon_{it}^{(r)}+o_P(1)\\&\xrightarrow{d}N(0,\Omega_r),\end{aligned}$$

where $\mathfrak{X}_{it}^{(r)}=(\mathfrak{X}_{1,it}^{(r)},...,\mathfrak{X}_{P,it}^{(r)})'$, $\tilde{B}_{2\beta}^{(r)}=(\tilde{B}_{2\beta,1}^{(r)},\cdots,\tilde{B}_{2\beta,P}^{(r)})'$, and the convergence in distribution follows from Assumption A.5. ■

Proof of Lemma A.9. (i) Note that

$$D(\hat{F}^{(r)})-D(F^{(r)})=\frac{1}{NT}\sum_{i=1}^N X_i^{(r)\prime}\left(M_{\hat{F}^{(r)}}-M_{F^{(r)}}\right)X_i^{(r)}-\frac{1}{T}\left[\frac{1}{N^2}\sum_{i=1}^N\sum_{j=1}^N X_i^{(r)\prime}(M_{\hat{F}^{(r)}}-M_{F^{(r)}})a_{ij}^{(r)}X_j^{(r)}\right].$$

It is easy to show that either term on the right hand side is bounded above by $O_P\left(\left\|P_{\hat{F}^{(r)}}-P_{F^{(r)}}\right\|\right)$, which is $O_P(C_{NT}^{-1}+\|\hat{\delta}_r\|^{1/2})$ by Lemma A.2(v). The result then follows from the asymptotic nonsingularity of $D(F^{(r)})$

(ii) Let $\tilde{G}^{(r)}=(\frac{1}{T}F^{(r)\prime}F^{(r)})^{-1}(\frac{1}{N}\Lambda_r'\Lambda_r)^{-1}$ and $\bar{G}^{(r)}=\left(\frac{1}{T}E\left[F^{(r)\prime}F^{(r)}\right]\right)^{-1}(\frac{1}{N}\Lambda_r'\Lambda_r)^{-1}$. We can readily show that

$$\xi_{NT}^{(r)}-B_{3\beta}^{(r)}=-\frac{1}{N}\sum_{i=1}^N\frac{(X_i^{(r)}-V_i^{(r)})'F^{(r)}}{T}\tilde{G}^{(r)}\frac{1}{T}\sum_{j=1}^N\sum_{t=1}^T k_{h,tr}^*\lambda_{jr}\left[\varepsilon_{it}\varepsilon_{jt}-E(\varepsilon_{it}\varepsilon_{jt})\right]=-\Phi^{(r)}+o_P((Th/N)^{-1/2}),$$

where $\Phi^{(r)}=\frac{1}{NT^2}\sum_{i=1}^N(X_i^{(r)}-V_i^{(r)})'F^{(r)}\bar{G}^{(r)}\sum_{j=1}^N\sum_{t=1}^T k_{h,tr}^*\lambda_{jr}\left[\varepsilon_{it}\varepsilon_{jt}-E_{\mathcal{C}}(\varepsilon_{it}\varepsilon_{jt})\right]$.

$$\begin{aligned}\Phi^{(r)}&=\frac{1}{NT^2}\sum_{i=1}^N X_i^{(r)\prime}F^{(r)}\bar{G}^{(r)}\sum_{j=1}^N\sum_{t=1}^T k_{h,tr}^*\lambda_{jr}\left[\varepsilon_{it}\varepsilon_{jt}-E_{\mathcal{C}}(\varepsilon_{it}\varepsilon_{jt})\right]\\&\quad-\frac{1}{NT^2}\sum_{i=1}^N V_i^{(r)\prime}F^{(r)}\bar{G}^{(r)}\sum_{j=1}^N\sum_{t=1}^T k_{h,tr}^*\lambda_{jr}\left[\varepsilon_{it}\varepsilon_{jt}-E_{\mathcal{C}}(\varepsilon_{it}\varepsilon_{jt})\right]\equiv\Phi_1^{(r)}-\Phi_2^{(r)}.\end{aligned}$$

For $\Phi_1^{(r)}$, we make further decomposition:

$$\begin{aligned}\Phi_1^{(r)}&=\frac{1}{NT^2}\sum_{i=1}^N\sum_{j=1}^N\sum_{t=1}^T\sum_{s=1}^T E_{\mathcal{C}}(X_{is}^{(r)}F_s^{(r)\prime})\bar{G}^{(r)}k_{h,tr}\lambda_{jr}\left[\varepsilon_{it}\varepsilon_{jt}-E_{\mathcal{C}}(\varepsilon_{it}\varepsilon_{jt})\right]\\&\quad+\frac{1}{NT^2}\sum_{i=1}^N\sum_{j=1}^N\sum_{t=1}^T\sum_{s=1}^T\left[X_{is}^{(r)}F_s^{(r)\prime}-E_{\mathcal{C}}(X_{is}^{(r)}F_s^{(r)\prime})\right]\bar{G}^{(r)}k_{h,tr}\lambda_{jr}\left[\varepsilon_{it}\varepsilon_{jt}-E_{\mathcal{C}}(\varepsilon_{it}\varepsilon_{jt})\right]\equiv\Phi_{1,1}^{(r)}-\Phi_{1,2}^{(r)}.\end{aligned}$$

We can show that $\text{Var}_{\mathcal{C}}(\Phi_{1,1}^{(r)})=O_P\left((Th)^{-1}\right)$, which, along with the fact that $E_{\mathcal{C}}(\Phi_{1,1}^{(r)})=0$, implies that $\Phi_{1,1}^{(r)}=O_P\left((Th)^{-1/2}\right)$. Similarly, for $\Phi_{1,2}^{(r)}$, we can show that $E_{\mathcal{C}}(\Phi_{1,2}^{(r)})=O_P(N/T)$, $\text{Var}_{\mathcal{C}}(\Phi_{1,2}^{(r)})=O_P\left((Th)^{-1}\right)$ and thus $\Phi_{1,2}^{(r)}=O_P\left(N/T+(Th)^{-1/2}\right)$. It follows that $\sqrt{Th/N}\Phi_1^{(r)}=O_P(N^{-1/2}+(Nh/T)^{-1/2})=o_P(1)$. Similarly, we can show that $\sqrt{Th/N}\Phi_2^{(r)}=o_P(1)$. Then $\sqrt{Th/N}\left[\xi_{NT}^{(r)}-B_{3\beta}^{(r)}\right]=o_P(1)$.

(iii) Denote $G^{(r)} = (\frac{1}{T}F^{(r)\prime}\hat{F}^{(r)})^{-1}(\frac{1}{N}\Lambda_r'\Lambda_r)^{-1}$ and $\tilde{G}^{(r)} = (\frac{1}{T}F^{(r)\prime}F^{(r)})^{-1}(\frac{1}{N}\Lambda_r'\Lambda_r)^{-1}$. Then

$$\begin{aligned}
\sqrt{\frac{N}{Th}}\left(\zeta_{NT}^{(r)} - B_{4\beta}^{(r)}\right) &= \frac{h^{1/2}}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}X_i^{(r)\prime}\left[M_{\hat{F}^{(r)}}\varepsilon_j^{(r)}\varepsilon_j^{(r)\prime}\hat{F}^{(r)}G^{(r)} - M_{F^{(r)}}\varepsilon_j^{(r)}\varepsilon_j^{(r)\prime}F^{(r)}\bar{G}^{(r)}\right]\lambda_{ir} \\
&= \frac{h^{1/2}}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}X_i^{(r)\prime}\left[M_{\hat{F}^{(r)}} - M_{F^{(r)}}\right]\varepsilon_j^{(r)}\varepsilon_j^{(r)\prime}\hat{F}^{(r)}G^{(r)}\lambda_{ir} \\
&\quad + \frac{h^{1/2}}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}X_i^{(r)\prime}M_{F^{(r)}}\varepsilon_j^{(r)}\varepsilon_j^{(r)\prime}\left[\hat{F}^{(r)}G^{(r)} - F^{(r)}\bar{G}^{(r)}\right]\lambda_{ir} \equiv III_1 + III_2.
\end{aligned}$$

For III_1, we have

$$\begin{aligned}
III_1 &= \frac{-h^{1/2}}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}X_i^{(r)\prime}\left[P_{\hat{F}^{(r)}} - P_{F^{(r)}}\right]\varepsilon_j^{(r)}\varepsilon_j^{(r)\prime}\hat{F}^{(r)}G^{(r)}\lambda_{ir} \\
&\leq \frac{h^{1/2}}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}X_i^{(r)\prime}\left[P_{\hat{F}^{(r)}} - P_{F^{(r)}}\right]\varepsilon_j^{(r)}\varepsilon_j^{(r)\prime}F^{(r)}H^{(r)}G^{(r)}\lambda_{ir} \\
&\quad + \frac{h^{1/2}}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}X_i^{(r)\prime}\left[P_{\hat{F}^{(r)}} - P_{F^{(r)}}\right]\varepsilon_j^{(r)}\varepsilon_j^{(r)\prime}[\hat{F}^{(r)} - F^{(r)}H^{(r)}]G^{(r)}\lambda_{ir} \equiv III_{1,1} + III_{1,2}.
\end{aligned}$$

For $III_{1,2}$, it suffices to consider the rough probability bound

$$\begin{aligned}
|\omega' III_{1,2}| &\leq \left\|\frac{h^{1/2}}{N^{3/2}T^{3/2}}\sum_{j=1}^{N}\varepsilon_j^{(r)\prime}[\hat{F}^{(r)} - F^{(r)}H^{(r)}]G^{(r)}\lambda_{ir}\omega'\sum_{i=1}^{N}X_i^{(r)\prime}\left[P_{\hat{F}^{(r)}} - P_{F^{(r)}}\right]\varepsilon_j^{(r)}\right\| \\
&\lesssim (NTh)^{1/2}\left\{\frac{1}{N}\sum_{j=1}^{N}\left\|\frac{1}{T}\varepsilon_j^{(r)\prime}[\hat{F}^{(r)} - F^{(r)}H^{(r)}]\right\|^2\right\}^{1/2}\left\{\frac{1}{N}\sum_{j=1}^{N}\left\|\frac{1}{NT}\sum_{i=1}^{N}X_i^{(r)\prime}\left[P_{\hat{F}^{(r)}} - P_{F^{(r)}}\right]\varepsilon_j^{(r)}\right\|^2\right\}^{1/2} \\
&\lesssim (NTh)^{1/2}\left\{\frac{1}{N}\sum_{j=1}^{N}\left\|\frac{1}{T}\varepsilon_j^{(r)\prime}[\hat{F}^{(r)} - F^{(r)}H^{(r)}]\right\|^2\right\}^{1/2}\left\|P_{\hat{F}^{(r)}} - P_{F^{(r)}}\right\| \\
&= (NTh)^{1/2}O_P\left(C_{NT}^{-2}\right)O_P\left(C_{NT}^{-1}\right) = o_P(1).
\end{aligned}$$

Using the decomposition

$$\begin{aligned}
P_{\hat{F}^{(r)}} - P_{F^{(r)}} &= \frac{1}{T}\hat{F}^{(r)}\hat{F}^{(r)\prime} - F^{(r)}[F^{(r)\prime}F^{(r)}]^{-1}F^{(r)\prime} \\
&= \frac{1}{T}\left(\hat{F}^{(r)} - F^{(r)}H^{(r)}\right)\hat{F}^{(r)\prime} + F^{(r)}H^{(r)}(\hat{F}^{(r)} - F^{(r)}H^{(r)})' \\
&\quad + \frac{1}{T}F^{(r)}H^{(r)}\left\{[H^{(r)\prime}\frac{1}{T}F^{(r)\prime}F^{(r)}H^{(r)}]^{-1} - \mathbb{I}_R\right\}H^{(r)\prime}F^{(r)\prime} \\
&= \frac{1}{T}\left(\hat{F}^{(r)} - F^{(r)}H^{(r)} - B^{(r)}\right)\hat{F}^{(r)\prime} + \frac{1}{T}F^{(r)}H^{(r)}(\hat{F}^{(r)} - F^{(r)}H^{(r)} - B^{(r)})' \\
&\quad + \frac{1}{T}B^{(r)}\hat{F}^{(r)\prime} + \frac{1}{T}F^{(r)}H^{(r)}B^{(r)\prime} + \frac{1}{T}F^{(r)}\left\{\frac{1}{T}[F^{(r)\prime}F^{(r)}]^{-1} - H^{(r)}H^{(r)\prime}\right\}F^{(r)\prime},
\end{aligned}$$

we can readily show that $\frac{1}{NT}\sum_{i=1}^{N}X_i^{(r)\prime}\left[P_{\hat{F}^{(r)}} - P_{F^{(r)}}\right]\varepsilon_j^{(r)} = O_P\left(C_{NT}^{-2}\right)$ for each j and $\frac{1}{N}\sum_{j=1}^{N}||\frac{1}{NT}\sum_{i=1}^{N}X_i^{(r)\prime}$

$\left[P_{\hat{F}^{(r)}}-P_{F^{(r)}}\right]\varepsilon_j^{(r)}||^2=O_P\left(C_{NT}^{-4}\right)$. Then

$$\begin{aligned}|\omega' III_{1,1}| &\leq \left\|\frac{h^{1/2}}{N^{3/2}T^{3/2}}\sum_{j=1}^{N}\varepsilon_j^{(r)\prime}F^{(r)}H^{(r)}]G^{(r)}\lambda_{ir}\omega'\sum_{i=1}^{N}X_i^{(r)\prime}\left[P_{\hat{F}^{(r)}}-P_{F^{(r)}}\right]\varepsilon_j^{(r)}\right\| \\ &\lesssim (NTh)^{1/2}\left\{\frac{1}{N}\sum_{j=1}^{N}\left\|\frac{1}{T}\varepsilon_j^{(r)\prime}F^{(r)}\right\|^2\right\}^{1/2}\left\{\frac{1}{N}\sum_{j=1}^{N}\left\|\frac{1}{NT}\sum_{i=1}^{N}X_i^{(r)\prime}\left[P_{\hat{F}^{(r)}}-P_{F^{(r)}}\right]\varepsilon_j^{(r)}\right\|^2\right\}^{1/2} \\ &= (NTh)^{1/2}O_P((Th)^{-1/2})O_P\left(C_{NT}^{-2}\right)=o_P(1).\end{aligned}$$

Then $III_1=o_P(1)$.

For III_2, we have

$$\begin{aligned}III_2 &= \frac{h^{1/2}}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}X_i^{(r)\prime}M_{F^{(r)}}\varepsilon_j^{(r)}\varepsilon_j^{(r)\prime}[\hat{F}^{(r)}(\frac{1}{T}F^{(r)\prime}\hat{F}^{(r)})^{-1}-F^{(r)}(\frac{1}{T}F^{(r)\prime}F^{(r)})^{-1}][\frac{1}{N}\Lambda^{(r)\prime}\Lambda^{(r)}]^{-1}\lambda_{ir} \\ &= \frac{h^{1/2}}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}X_i^{(r)\prime}\varepsilon_j^{(r)}\varepsilon_j^{(r)\prime}[\hat{F}^{(r)}-F^{(r)}H^{(r)}](\frac{1}{T}F^{(r)\prime}\hat{F}^{(r)})^{-1}[\frac{1}{N}\Lambda^{(r)\prime}\Lambda^{(r)}]^{-1}\lambda_{ir} \\ &+ \frac{h^{1/2}}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}X_i^{(r)\prime}\varepsilon_j^{(r)}\varepsilon_j^{(r)\prime}F^{(r)}H^{(r)}[(\frac{1}{T}F^{(r)\prime}\hat{F}^{(r)})^{-1}-(\frac{1}{T}F^{(r)\prime}F^{(r)}H^{(r)})^{-1}][\frac{1}{N}\Lambda^{(r)\prime}\Lambda^{(r)}]^{-1}\lambda_{ir} \\ &- \frac{h^{1/2}}{N^{3/2}T^{35/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}X_i^{(r)\prime}F^{(r)}[\frac{1}{T}F^{(r)\prime}F^{(r)}]^{-1}F^{(r)\prime}\varepsilon_j^{(r)}\varepsilon_j^{(r)\prime}[\hat{F}^{(r)}-F^{(r)}H^{(r)}](\frac{1}{T}F^{(r)\prime}\hat{F}^{(r)})^{-1}[\frac{1}{N}\Lambda^{(r)\prime}\Lambda^{(r)}]^{-1}\lambda_{ir} \\ &- \frac{h^{1/2}}{N^{3/2}T^{35/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}X_i^{(r)\prime}F^{(r)}[\frac{1}{T}F^{(r)\prime}F^{(r)}]^{-1}F^{(r)\prime}\varepsilon_j^{(r)}\varepsilon_j^{(r)\prime}F^{(r)}H^{(r)}[(\frac{1}{T}F^{(r)\prime}\hat{F}^{(r)})^{-1}-(\frac{1}{T}F^{(r)\prime}F^{(r)}H^{(r)})^{-1}] \\ &\times[\frac{1}{N}\Lambda^{(r)\prime}\Lambda^{(r)}]^{-1}\lambda_{ir} \\ &\equiv III_{2,1}+III_{2,2}-III_{2,3}-III_{2,4}.\end{aligned}$$

We can readily show that $III_{2,l}=o_P(1)$ for $l\in[4]$. For example,

$$\begin{aligned}|\omega' III_{2,1}| &= \frac{h^{1/2}}{N^{3/2}T^{3/2}}\left|\sum_{j=1}^{N}\varepsilon_j^{(r)\prime}\left[\hat{F}^{(r)}-F^{(r)}H^{(r)}\right](\frac{1}{T}F^{(r)\prime}\hat{F}^{(r)})^{-1}[\frac{1}{N}\Lambda^{(r)\prime}\Lambda^{(r)}]^{-1}\sum_{i=1}^{N}\lambda_{ir}\omega' X_i^{(r)\prime}\varepsilon_j^{(r)}\right| \\ &\lesssim (NTh)^{1/2}\left\{\frac{1}{N}\sum_{j=1}^{N}\left\|\frac{1}{T}\varepsilon_j^{(r)\prime}\left[\hat{F}^{(r)}-F^{(r)}H^{(r)}\right]\right\|^2\right\}^{1/2}\left\{\frac{1}{N}\sum_{j=1}^{N}\left\|\frac{1}{NT}\sum_{i=1}^{N}\lambda_{ir}\omega' X_i^{(r)\prime}\varepsilon_j^{(r)}\right\|^2\right\}^{1/2} \\ &= (NTh)^{1/2}O(C_{NT}^{-2})O_P((NTh)^{-1/2})=o_P(1).\end{aligned}$$

This prove the lemma. ■

Proof of Lemma A.10 (i) Recall that $\tilde{\mathbf{X}}_p^{(r)}=\mathbf{X}_p^{(r)}-E_{\mathcal{C}}(\mathbf{X}_p^{(r)})$. Let $\tilde{X}_{p,is}^{(r)}$ denote the (s,i)th element of the $T\times N$ matrix $\tilde{\mathbf{X}}_p^{(r)}$. Let e_{qR} denotes the qth column of $\mathbb{I}_R$. Then

$$\left\|\frac{\sqrt{NTh}}{NT}\operatorname{tr}\left\{P_{F^{(r)}}\left[\tilde{\mathbf{X}}_p^{(r)}\varepsilon^{(r)\prime}-E_{\mathcal{C}}(\tilde{\mathbf{X}}_p^{(r)}\varepsilon^{(r)\prime})\right]\right\}\right\|$$

$$= \left\| \sum_{l,q=1}^{R} e'_{lR} (\frac{1}{T} F^{(r)\prime} F^{(r)})^{-1} e_{qR} \frac{\sqrt{h}}{\sqrt{N} T^{3/2}} \sum_{i=1}^{N} \sum_{t,s=1}^{T} F_{t,l}^{(r)} F_{s,q}^{(r)} \left[\tilde{X}_{p,is}^{(r)} \varepsilon_{it}^{(r)} - E_{\mathcal{C}} (\tilde{X}_{p,is}^{(r)} \varepsilon_{it}^{(r)}) \right] \right\|$$

$$\lesssim (Th)^{-1/2} \sum_{l,q=1}^{R} \|\Xi_{NT,lq}\| = O_P((Th)^{-1/2}),$$

where $\Xi_{NT,lq} = \frac{h}{N^{1/2}T} \sum_{i=1}^{N} \sum_{t,s=1}^{T} k_{h,tr}^{*} k_{h,sr}^{*} F_{t,l} F_{s,q} [\tilde{X}_{p,is} \varepsilon_{it} - E_{\mathcal{C}}(\tilde{X}_{p,is} \varepsilon_{it})] = O_P(1)$ by Assumption A.2(iv).

(ii) Recall $a_{ij}^{(r)} = \lambda'_{ir} (\frac{1}{N} \Lambda'_r \Lambda_r)^{-1} \lambda_{jr}$. Then

$$\begin{aligned}
\frac{\sqrt{NTh}}{NT} \text{tr} \left(\tilde{\mathbf{X}}_k^{(r)} P_{\Lambda_r} \varepsilon^{(r)\prime} \right) &= \frac{\sqrt{NTh}}{N^2 T} \sum_{t=1}^{T} \sum_{i=1}^{N} \sum_{j=1}^{N} k_{h,tr}^{*} a_{ij}^{(r)} \tilde{X}_{k,jt} \varepsilon_{it} \\
&= \frac{\sqrt{NTh}}{N^2 T} \sum_{t=1}^{T} \sum_{i=1}^{N} k_{h,tr}^{*} a_{ii}^{(r)} \tilde{X}_{k,it} \varepsilon_{it} + \frac{\sqrt{NTh}}{N^2 T} \sum_{t=1}^{T} \sum_{1 \le i < j \le N} k_{h,tr}^{*} a_{ij}^{(r)} \tilde{X}_{k,jt} \varepsilon_{it} \\
&\quad + \frac{\sqrt{NTh}}{N^2 T} \sum_{t=1}^{T} \sum_{1 \le j < i \le N} k_{h,tr}^{*} a_{ij}^{(r)} \tilde{X}_{k,jt} \varepsilon_{it} \equiv IV_1 + IV_2 + IV_3.
\end{aligned}$$

It is easy to show that $IV_1 = O_P\left(N^{-1/2}\right)$. For IV_2, we have $E_{\mathcal{C}}\left(IV_2\right) = 0$ and

$$\text{Var}_{\mathcal{C}}(\omega' IV_2) = \frac{h}{N^3 T} \sum_{t=1}^{T} \sum_{s=1}^{T} \sum_{1 \le i < j \le N} k_{h,tr}^{*} k_{h,sr}^{*} \left(a_{ij}^{(r)} \right)^2 \omega' E_{\mathcal{C}} \left[\tilde{X}_{k,jt} \varepsilon_{it} \varepsilon_{is} \tilde{X}'_{k,js} \right] \omega = O_P\left(N^{-1}\right).$$

Then $IV_2 = O_P\left(N^{-1/2}\right)$. By the same token, $IV_3 = O_P\left(N^{-1/2}\right)$. It follows that $\frac{\sqrt{NTh}}{NT} \text{tr}(\tilde{\mathbf{X}}_k^{(r)} P_{\Lambda_r} \varepsilon^{(r)\prime}) = o_P\left(1\right)$.

(iii) Let $\lambda_{ir,q}$ denote the qth element in λ_{ir}. Noting that

$$\begin{aligned}
E_{\mathcal{C}} \left\| \tilde{X}_k^{(r)} \Lambda_r \right\|^2 &= \sum_{q,l=1}^{R} \sum_{i,j=1}^{N} \sum_{t,s=1}^{T} k_{h,tr}^{*1/2} k_{h,sr}^{*1/2} \lambda_{ir,q} \lambda_{jr,l} \text{Cov}_{\mathcal{C}}\left(X_{k,it}, X_{k,js}\right) \\
&\lesssim \sum_{i,j=1}^{N} \sum_{t,s=1}^{T} \left(k_{h,tr}^{*} + k_{h,sr}^{*} \right) \left| \text{Cov}_{\mathcal{C}}\left(X_{k,it}, X_{k,js}\right) \right| = 2 \sum_{t=1}^{T} k_{h,tr}^{*} \sum_{i,j=1}^{N} \sum_{s=1}^{T} \left| \text{Cov}_{\mathcal{C}}\left(X_{k,it}, X_{k,js}\right) \right| \\
&\le \max_{s} \sum_{i,j=1}^{N} \sum_{s=1}^{T} \left| \text{Cov}_{\mathcal{C}}\left(X_{k,it}, X_{k,js}\right) \right| \sum_{t=1}^{T} k_{h,tr}^{*} = O_P(N) O_P(T),
\end{aligned}$$

$\left\| \tilde{X}_k^{(r)} \Lambda_r \right\| = O_P\left((NT)^{1/2}\right)$ by the conditional Chebyshev inequality. Similarly, in view of the fact that

$$\begin{aligned}
E \left\| \Lambda'_r \varepsilon^{(r)\prime} F^{(r)} \right\|^2 &= \sum_{i,j=1}^{N} \sum_{t,s=1}^{T} E \left[\varepsilon_{it}^{(r)} \varepsilon_{js}^{(r)} F_t^{(r)\prime} \lambda_{ir} \lambda'_{jr} F_s^{(r)} \right] = \sum_{i,j=1}^{N} \sum_{t,s=1}^{T} k_{h,tr}^{*} k_{h,sr}^{*} \lambda'_{r,i} E\left[F_t \varepsilon_{it} \varepsilon_{js} F'_s\right] \lambda_{r,j} \\
&\lesssim \sum_{i,j=1}^{N} \sum_{t,s=1}^{T} \left(k_{h,tr}^{*2} + k_{h,sr}^{*2} \right) \left\| E\left[F_t \varepsilon_{it} \varepsilon_{js} F'_s\right] \right\| = 2 \max_{t} \sum_{i,j=1}^{N} \sum_{s=1}^{T} \left\| E\left[F_t \varepsilon_{it} \varepsilon_{js} F'_s\right] \right\| \sum_{t=1}^{T} k_{h,tr}^{*2} \\
&= O(N) O_P(Th^{-1}) = O_P(NTh^{-1}),
\end{aligned}$$

we have $\left\|\Lambda_r'\varepsilon^{(r)\prime}F^{(r)}\right\| = O_P((NTh^{-1})^{1/2})$. Then

$$\begin{aligned}\left|\frac{\sqrt{NTh}}{NT}\text{tr}\left(P_{F^{(r)}}\tilde{\mathbf{X}}_k^{(r)}P_{\Lambda_r}\varepsilon^{(r)\prime}\right)\right| &= \left|\frac{\sqrt{NTh}}{NT}\text{tr}\left((F^{(r)\prime}F^{(r)})^{-1}F^{(r)\prime}\tilde{\mathbf{X}}_k^{(r)}\Lambda_r\left(\Lambda_r'\Lambda_r\right)^{-1}\Lambda_r'\varepsilon^{(r)\prime}F^{(r)}\right)\right| \\ &\leq \frac{\sqrt{NTh}}{NT}\left\|(F^{(r)\prime}F^{(r)})^{-1}F^{(r)\prime}\right\|\left\|\tilde{\mathbf{X}}_k^{(r)}\Lambda_r\right\|\left\|(\Lambda_r'\Lambda_r)^{-1}\right\|\left\|\Lambda_r'\varepsilon^{(r)\prime}F^{(r)}\right\| \\ &= \frac{\sqrt{NTh}}{NT}O_P(T^{-1/2})O_P((NTh^{-1})^{1/2})O_P(N^{-1})O_P((NT/h)^{1/2}) \\ &= O_P((Nh)^{-1/2}) = o_P(1). \blacksquare\end{aligned}$$

Proof of Lemma A.11. (i) Following the proof of Lemma A.7(ii) in Su and Wang (2017), we can show that uniformly in r,

$$\begin{aligned}H^{(r)} &= \left(N^{-1}\Lambda_r'\Lambda_r\right)(T^{-1}F^{(r)\prime}\hat{F}^{(r)})\hat{V}_{NT}^{(r)-1} = \Sigma_{\Lambda_r}^{1/2}\Upsilon_r V_r^{-1/2} + o_P(C_{NT}^{-1}(\ln T)^{1/2}) + O(\max_r\|\hat{\beta}_r - \beta_r\|) \\ &= \Sigma_{\Lambda_r}^{1/2}\Upsilon_r V_r^{-1/2} + o_P(C_{NT}^{-1}(\ln T)^{1/2}) \equiv Q_r + o_P(C_{NT}^{-1}(\ln T)^{1/2}),\end{aligned}$$

where the third equality follows from the fact $\max_r\|\hat{\beta}_r - \beta_r\| = O_P((NTh)^{-1/2}(\ln T)^{1/2})$ by Theorem 3.4. Following the proof of Lemma A.7(iii)-(iv) in Su and Wang (2020), we can show that

$$\begin{aligned}\max_r\frac{1}{T}\left\|\hat{F}^{(r)} - F^{(r)}H^{(r)} - B^{(r)}\right\|^2 &= O_P(T^{-1}h^{-1} + N^{-1}\ln T) + O_P(\max_r\|\hat{\beta}_r - \beta_r\|), \text{ and} \\ \max_r\frac{1}{T}\left\|(\hat{F}^{(r)} - F^{(r)}H^{(r)} - B^{(r)})'F^{(r)}\right\| &= O_P(T^{-1}h^{-1} + N^{-1}\ln T) + O_P(\max_r\|\hat{\beta}_r - \beta_r\|),\end{aligned}$$

which in conjunction with the fact that $\frac{1}{T}\left\|B^{(r)}\right\|^2 = O_P\left(h^2\right)$, $\frac{1}{T}\left\|B^{(r)\prime}F^{(r)}\right\| = O_P\left(h^2\right)$ and $\max_r\|\hat{\beta}_r - \beta_r\| = O_P((NTh)^{-1/2}(\ln T)^{1/2})$, implies that

$$\begin{aligned}\max_r\frac{1}{T}\left\|\hat{F}^{(r)} - F^{(r)}H^{(r)}\right\|^2 &= O_P(h^2 + T^{-1}h^{-1} + N^{-1}\ln T) \text{ and} \\ \max_r\frac{1}{T}\left\|(\hat{F}^{(r)} - F^{(r)}H^{(r)} - B^{(r)})'F^{(r)}\right\| &= O_P(h^2 + T^{-1}h^{-1} + N^{-1}\ln T).\end{aligned}$$

Then using that $\frac{1}{T}\hat{F}^{(r)\prime}\hat{F}^{(r)} = \mathbb{I}_R$, we have uniformly in r

$$\begin{aligned}&\left\|\left(H^{(r)}H^{(r)\prime}\right)^{-1} - \frac{1}{T}F^{(r)\prime}F^{(r)}\right\| \\ =\ &\left\|H^{(r)\prime-1}\left[\frac{1}{T}\hat{F}^{(r)\prime}\hat{F}^{(r)} - \frac{1}{T}H^{(r)\prime}F^{(r)\prime}F^{(r)}H^{(r)}\right]H^{(r)-1}\right\| \\ \leq\ &\left\|H^{(r)\prime-1}\right\|^2\left\|\frac{1}{T}(\hat{F}^{(r)} - F^{(r)}H^{(r)})'(\hat{F}^{(r)} - F^{(r)}H^{(r)})\right\| + 2\left\|H^{(r)\prime-1}\right\|\left\|\frac{1}{T}(\hat{F}^{(r)} - F^{(r)}H^{(r)})'F^{(r)}\right\| \\ =\ &O_P(h^2 + T^{-1}h^{-1} + N^{-1}\ln T).\end{aligned}$$

Consequently, $\max_r\left\|H^{(r)}H^{(r)\prime} - (\frac{1}{T}F^{(r)\prime}F^{(r)})^{-1}\right\| = O_P(h^2 + T^{-1}h^{-1} + N^{-1}\ln T)$.

(ii) Using $A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$, we have

$$\begin{aligned}&\max_t\left\|(\frac{1}{N}\hat{\Lambda}_t'\hat{\Lambda}_t)^{-1} - H^{(t)\prime}(\frac{1}{N}\Lambda_t'\Lambda_t)^{-1}H^{(t)}\right\| \\ =&\left\|(\frac{1}{N}\hat{\Lambda}_t'\hat{\Lambda}_t)^{-1}\left[H^{(t)-1}(\frac{1}{N}\Lambda_t'\Lambda_t)H^{(t)\prime-1} - \frac{1}{N}\hat{\Lambda}_t'\hat{\Lambda}_t\right]H^{(t)\prime}(\frac{1}{N}\Lambda_t'\Lambda_t)^{-1}H^{(t)}\right\|\end{aligned}$$

$$\leq \max_t \left\| (\frac{1}{N}\hat{\Lambda}_t'\hat{\Lambda}_t)^{-1} \right\| \left\| H^{(t)-1}(\frac{1}{N}\Lambda_t'\Lambda_t)H^{(t)\prime -1} - \frac{1}{N}\hat{\Lambda}_t'\hat{\Lambda}_t \right\| \left\| H^{(t)\prime}(\frac{1}{N}\Lambda_t'\Lambda_t)^{-1}H^{(t)} \right\|$$

Note that $\max_t \left\| (\frac{1}{N}\hat{\Lambda}_t'\hat{\Lambda}_t)^{-1} \right\| = O_P(1)$ and $\max_t \left\| H^{(t)\prime}(\frac{1}{N}\Lambda_t'\Lambda_t)^{-1}H^{(t)} \right\| = O_P(1)$. It remains to study

$$\max_t \left\| \frac{1}{N}\hat{\Lambda}_t'\hat{\Lambda}_t - H^{(t)-1}(\frac{1}{N}\Lambda_t'\Lambda_t)H^{(t)\prime -1} \right\| \leq \max_t \frac{1}{N} \left\| \hat{\Lambda}_t - \Lambda_t H^{(t)\prime -1} \right\|^2 + 2\max_t \frac{1}{N} \left\| (\hat{\Lambda}_t - \Lambda_t H^{(t)\prime -1})'\Lambda_t H^{(t)\prime -1} \right\|.$$

Following the proof of Lemma A.6(xii) and (xiii) in Su and Wang (2020), and the proof of Theorem 3.3(ii), we can show that

$$\max_t \frac{1}{N} \left\| \hat{\Lambda}_t - \Lambda_t H^{(t)\prime -1} - B_\Lambda^{(t)} \right\|^2 = O_P(C_{NT}^{-2}\ln T) + O(\max_t \|\hat{\beta}_t - \beta_t\|) \text{ and}$$
$$\max_t \frac{1}{N} \left\| (\hat{\Lambda}_t - \Lambda_t H^{(t)\prime -1} - B_\Lambda^{(t)})'\Lambda_t H^{(t)\prime -1} \right\| = O_P(C_{NT}^{-2}\ln T) + O(\max_t \|\hat{\beta}_t - \beta_t\|).$$

In addition, $\max_t \frac{1}{N} \left\| B_\Lambda^{(t)} \right\|^2 = O_P(h^4)$ and $\max_t \frac{1}{N} \left\| B_\Lambda^{(t)\prime}\Lambda_t H^{(t)\prime -1} \right\| = O_P(h^2)$. Then

$$\max_t \left\| (\frac{1}{N}\hat{\Lambda}_t'\hat{\Lambda}_t)^{-1} - H^{(t)\prime}(\frac{1}{N}\Lambda_t'\Lambda_t)^{-1}H^{(t)} \right\| = O_P(h^2 + C_{NT}^{-2}\ln T) + O_P(\max_t \|\hat{\beta}_t - \beta_t\|) = O_P(h^2 + C_{NT}^{-2}\ln T). \blacksquare$$

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