Problem Set I for Econometrics

Definitions and Notations

- Vector space. A nonempty set $X \subset \mathbb{R}^n$ is a vector space if $x + y \in X$ and $cx \in X$ for all $x, y \in X$ and scalar c.
- Span. The span of a set of vectors is the set of all linear combinations of the vectors. For example, the x-y plane is spanned by $(1, 0)$ and $(0, 1)$.
- Range. Given a matrix A, the range of A is defined as the span of its columns,

$$\mathcal{R}(A) = \{y | y = Ax, \text{for some } x\}.$$

- Orthogonal complement of $\mathcal{R}(A)$, denoted by $\mathcal{R}(A)^\perp$, is defined by

$$\mathcal{R}(A)^\perp = \{x | x'y = 0 \text{ for all } y \in \mathcal{R}(A)\}.$$

- Null space. The null space of A is the set of all column vectors x such that $Ax = 0$,

$$\mathcal{N}(A) = \{x | Ax = 0\}.$$

- Basis. An independent subset of a vector space X that spans X is called a basis of X. Independence here means that any vector in the set cannot be written as a linear combination of other vectors in the set.
- Dimension of a vector space. The dimension of a vector space is the number of vectors in a basis of X.
- Projection. A matrix P is idempotent if $P^2 = P$. Idempotent matrices are often called projection matrices or projections. An orthogonal projection P is a projection such that, for all $x \in X$ and $y \in \mathcal{R}(P)$, $(x - Px)'y = 0$.
- Positive semi-definiteness (p.s.d.). We denote $A \geq 0$ if A is positive semi-definite, that is, $x'Ax \geq 0$. for all x. We denote $A \geq B$ if $A - B$ is p.s.d..
- Let iff denote "if and only if".

Problems

1 For any matrix A, show that $\mathcal{R}(A)^\perp = \mathcal{N}(A')$.

2 Let

$$A = \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{pmatrix},$$

Calculate the following

(a) The eigenvalues and eigenvectors of A

(b) A^{-1}, $\sqrt{A}$, and $\log(A)$

3 Show that

(a) If $A \geq 0$, then $B'AB \geq 0$ for all matrix B.

(b) If $A \geq B$, then $C'AC \geq C'BC$ for all matrix C.

(c) If all eigenvalues of a symmetric matrix A are non-negative, then $A \geq 0$.

4 Let A be an n-by-m matrix with independent columns, let P be the orthogonal projection on $\mathcal{R}(A)$. Show that

(a) $P = A(A'A)^{-1}A'$

(b) The eigenvalue of P is either 1 or 0.

(c) $I - P \geq 0$.