Math Tools

1 Total Differential

Consider a multivariate function $F(x_1, \ldots, x_n)$. The total differential of F at the point $(x_1^*, \ldots, x_n^*)$ is given by

$$dF = \frac{\partial F}{\partial x_1}(x_1^*, \ldots, x_n^*)dx_1 + \cdots + \frac{\partial F}{\partial x_n}(x_1^*, \ldots, x_n^*)dx_n.$$

The total differential characterizes the change of F at the point $(x_1^*, \ldots, x_n^*)$ using partial derivatives. To us, the total differential is useful in obtaining a *linear representation* of a change in a nonlinear multivariate function.

For example, consider an aggregate production function $Y = F(K, L)$ and an economy at the point (K^*, L^*). Then ΔY (a change in Y), if it is small compared to Y, can be well approximated by,

$$\Delta Y = \frac{\partial F}{\partial K}(K^*, L^*)\Delta K + \frac{\partial F}{\partial L}(K^*, L^*)\Delta L.$$

That is, the change in output has to come from the change in inputs, given the marginal product of capital and labor.

Often we denote the partial derivative $\frac{\partial F}{\partial x_i}(x_1^*, \ldots, x_n^*)$ by $F_i(x_1^*, \ldots, x_n^*)$, meaning the partial derivative of F with respect to the i-th argument. Or even simpler, we may omit the arguments and write F_i, when there is no confusion over the point where the total differential is taken. Using the shorthand notation, we may write the total differential formula as,

$$dF = F_1 dx_1 + \cdots + F_n dx_n.$$

2 Implicit Function Theorem

The implicit function theorem is very useful for the analysis of single-equation models. First consider the simplest case involving an equation of two variables, x and y:

Implicit Function Theorem: Let $G(x, y)$ be a differentiable function around (x^*, y^*). Suppose that $G(x^*, y^*) = 0$ and consider the equation $G(x, y) = 0$. If $\frac{\partial G}{\partial y}(x^*, y^*) \neq 0$, then there exists a differentiable function $y(x)$ defined on an interval around x^* such that $G(x, y(x)) = 0$ for all $x \in I$, $y(x^*) = y^*$, and

$$y'(x^*) \equiv \frac{dy}{dx}|_{x=x^*} = -\frac{\frac{\partial G}{\partial x}(x^*, y^*)}{\frac{\partial G}{\partial y}(x^*, y^*)}.$$

If $G(x, y) = 0$ represents a model, x is an exogenous variable, and y is an endogenous variable, then we can use the theorem to calculate the effect of a change in x on y.

And the theorem can be easily extended to accommodate more variables. If the model is $G(x_1, \ldots, x_n, y) = 0$, then it defines an implicit function $y = y(x_1, \ldots, x_n)$, and

$$\frac{\partial y}{\partial x_i}|_{x_1=x_1^*,\ldots,x_n=x_n^*,y=y^*} = -\frac{\frac{\partial G}{\partial x_i}(x_1 = x_1^*, \ldots, x_n = x_n^*, y = y^*)}{\frac{\partial G}{\partial y}(x_1 = x_1^*, \ldots, x_n = x_n^*, y = y^*}.$$

Or using the shorthand notation,

$$\frac{\partial y}{\partial x_i} = -\frac{G_i}{G_y},$$

where G_i represents partial derivative of G with respect to the i-the argument, which is x_i, and G_y is the partial derivative of G with respect to y.

3 Dealing with Multiple Equations

Implicit functions may be defined by multiple equations, or a set of equations. For example, the following model, which is a set of equations

$$\begin{aligned} F(x_1, x_2, y, z) &= 0 \\ G(x_1, x_2, y, z) &= 0 \end{aligned}$$

defines an implicit function $y = y(x_1, x_2)$. Here, we may understand that x_1 and x_2 are exogenous variables, and y and z are endogenous variables. To analyze the effects of x_1 and x_2 on y, for example, we need to calculate $\partial y/\partial x_1$ and $\partial y/\partial x_2$. For this purpose, there is a multiple-equation version of the implicit function theorem, but it is difficult to remember. Instead, we may first linearize the model using the total differential and apply the Cramer's rule to calculate the partial effects.

First, we apply the total differential to each equation and obtain,

$$\begin{aligned} F_1dx_1 + F_2dx_2 + F_ydy + F_zdz &= 0 \\ G_1dx_1 + G_2dx_2 + G_ydy + G_zdz &= 0 \end{aligned}$$

Now we have a linearized model. Moving all exogenous variables to the right hand side, we may represent the model as

$$\begin{bmatrix} F_y & F_z \\ G_y & G_z \end{bmatrix} \begin{bmatrix} dy \\ dz \end{bmatrix} = \begin{bmatrix} -F_1dx_1 - F_2dx_2 \\ -G_1dx_1 - G_2dx_2 \end{bmatrix},$$

which is a linear system of equations with dy and dz as unknown variables. Then we can apply Cramer's rule to obtain $\partial y/\partial x_1$, $\partial y/\partial x_2$, $\partial z/\partial x_1$, and $\partial z/\partial x_2$. For example, if we want to calculate $\partial y/\partial x_1$, then we force $dx_2 = 0$, meaning that x_2 is fixed. The Cramer's rule has

$$dy = \frac{\begin{vmatrix} -F_1 dx_1 - F_2 dx_2 & F_z \\ -G_1 dx_1 - G_2 dx_2 & G_z \end{vmatrix}}{\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}} = \frac{\begin{vmatrix} -F_1 & F_z \\ -G_1 & G_z \end{vmatrix}}{\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}} dx_1.$$

Hence

$$\frac{dy}{dx_1} = \frac{\begin{vmatrix} -F_1 & F_z \\ -G_1 & G_z \end{vmatrix}}{\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}}.$$

Note that since we force $dx_2 = 0$, the derivative $\frac{dy}{dx_1}$ is in fact partial derivative $\partial y/\partial x_1$. Similarly, we can calculate $\partial y/\partial x_2$, $\partial z/\partial x_1$, and $\partial z/\partial x_2$. And the method can be extended to deal with higher dimensional models.

Note that the Cramer's rule is formally stated as follows.

Cramer's Rule: Consider an n-dimensional system of linear equations, $Ax = b$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Then the solutions to x_1, x_2, etc. is given by

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}}{A}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} & \cdots & a_{1n} \\ a_{21} & b_2 & a_{23} & \cdots & a_{2n} \\ \vdots & & & \ddots & \vdots \\ a_{n1} & b_n & a_{n3} & \cdots & a_{nn} \end{vmatrix}}{A} \quad \cdots$$

In other words, the solution to x_i is given by a fraction. On the denominator is the determinant of A. On the numerator is the determinant of A with its i-th column replaced by the vector b.