Shrinkage Estimation of Common Breaks in Panel Data Models via Adaptive Group Fused Lasso*

Junhui Qian  
Antai College of Economics and Management, Shanghai Jiao Tong University  
Liangjun Su  
School of Economics, Singapore Management University  
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Abstract

In this paper we consider estimation and inference of common breaks in panel data models via adaptive group fused lasso. We consider two approaches – penalized least squares (PLS) for first-differenced models without endogenous regressors, and penalized GMM (PGMM) for first-differenced models with endogeneity. We show that with probability tending to one both methods can correctly determine the unknown number of breaks and estimate the common break dates consistently. We obtain estimates of the regression coefficients via post Lasso and establish their asymptotic distributions. We also propose and validate a data-driven method to determine the tuning parameter used in the Lasso procedure. Monte Carlo simulations demonstrate that both the PLS and PGMM estimation methods work well in finite samples. We apply our PGMM method to study the effect of foreign direct investment (FDI) on economic growth using a panel of 88 countries and regions from 1973 to 2012 and find multiple breaks in the model.

JEL Classification: C13, C23, C33, C51

Key Words: Adaptive Lasso; Change point; Group Lasso; Fused Lasso; Panel data; Penalized least squares; Penalized GMM; Structural change

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1 Introduction

Recently there has been a growing literature on the estimation and tests of common breaks in panel data models in which there are $N$ individual units and $T$ time series observations for each individual. Depending on whether $T$ is allowed to pass to infinity, the model is called “short” for fixed $T$ and “large” (or of large dimension) if $T$ passes to infinity. Implicitly, one usually allows $N$ to pass to infinity in panel data models.¹ Most of the literature falls into two categories depending on whether the parameters of interest are allowed to be heterogenous across individuals or not. The first category focuses on homogenous panel data models and includes De Watcher and Tzavalis (2005), Baltagi et al. (2012), and De Watcher and Tzavalis (2012). De Watcher and Tzavalis (2005) compare the relative performance of two model and moment selection methods in detecting breaks in short panels; Baltagi et al. (2012) consider the estimation and identification of change points in large dimensional panel models with either stationary or nonstationary regressors and error terms; De Watcher and Tzavalis (2012) develop a testing procedure for common breaks in short linear dynamic panel data models. The second category considers estimation and inference of common breaks in heterogenous panel data models; see Bai (2010), Kim (2011, 2012), Hsu and Lin (2012), Baltagi et al. (2013), among others. Bai (2010) establishes the asymptotic properties of the estimated break point in a location-scale heterogenous panel data model with either fixed or large $T$; Kim (2011) extends Bai’s (2010) method and develops an estimation procedure for a common deterministic time trend break in large heterogenous panels with a multi-factor error structure; Kim (2012) continues the study by estimating the common break date and common factors jointly; Hsu and Lin (2012) extends Bai’s (2010) theory to nonstationary panel data models where the error terms follow an I(1) process; Baltagi et al. (2013) study the estimation of large dimensional static heterogenous panels with a common break by extending Pesaran’s (2006) common correlated effects (CCE) estimation procedure. In addition, Chan et al. (2008) extend the testing procedure of Andrews (2003) from time series to heterogenous panels where the breaks may occur at different time points across individuals; Liao and Wang (2012) study the estimation of individual-specific structural breaks that exhibit a common distribution in a location-scale panel data model; Yamazaki and Kurozumi (2013) develop an LM-type test for slope homogeneity along the time dimension in fixed-effects panel data models with fixed $N$ and large $T$.²

A common feature of all of the above works is that a one-time break, common or not, is assumed in the estimation procedure. Although the assumption of a single break greatly facilitates the estimation and inference procedure, inferences based on it could be misleading if the underlying model has an unknown number of multiple breaks. For this reason, a large literature on the estimation and inference of models with multiple structural changes has been developed in the single or multiple time series framework; see,

¹Bai (1997a), Bai et al. (1998) and Qu and Perron (2007) extend the estimation of single-time series models to multiple-ones with simultaneous structural breaks where the number of equations is fixed.
²Pesaran and Yamagata (2008) and Su and Chen (2013) propose LM-type tests for slope homogeneity along the cross section dimension in large dimensional linear panel data models with additive fixed effects and interactive fixed effects, respectively.
e.g., Bai (1997a, 1997b), Bai and Perron (1998), Qu and Perron (2007), Su and White (2010), Kurozumi (2012), and Qian and Su (2013). In view of the fact that the conventional $avg$- and $exp$-type test statistics for multiple structural changes requires all permissible partitions of the sample which could be prohibitively large, Qian and Su (2013) propose shrinkage estimation of regression models with multiple structural changes by extending the fused Lasso of Tibshirani et al. (2005) to the time series regression framework.

In this paper we propose a shrinkage-based methodology for estimating panel data models with an unknown number of structural changes. The new methodology is most suitable for the vision that the regression coefficients in a panel data model may be time-varying but at the same time exhibit certain sparseness in abrupt changes or breaks. This vision seems pertinent in many applied studies using panel data that have a long time span measured in decades. During such a long time span, shocks to technologies, preferences, policies, and so on, may result in the change of a statistical relation applied economists seek to discover; but the shocks tend to be small over a relatively short time interval so that it does not alter the statistical relationship in short time. In this case, one has to allow the parameters in the model to change over time in an unknown way and recognize that parameters do not always alter from one time period to another one. Multiple structural breaks may occur during the whole time span but the number of breaks is generally small in comparison with the total number of time periods in the data, resulting in the sparseness of the breaks.

In terms of econometrics methodology, this paper extends the Lasso-type shrinkage approach in Qian and Su (2013) to panel data settings. To the best of our knowledge, this is the first in the literature to deal with panel data models with possibly multiple structural changes explicitly. To stay focused, we consider homogenous linear panel data models with an unknown number of common breaks and we do not allow cross section dependence. The extension to heterogenous panel data models and to panel data models with cross section dependence will be discussed at the end of Section 7. For the advantage of the use of panel data to study common breaks, we refer the readers directly to Bai (2010) and De Watcher and Tzavalis (2012). Despite the fact that the Lasso-type shrinkage estimation has a long history and wide applications in statistics (see, e.g., Tibshirani (1996), Knight and Fu (2000), Fan and Li (2001)), the application of Lasso-type shrinkage techniques in econometrics has a relatively short history. But the number of applications in econometrics has been increasing very fast in the last few years. For example, Caner (2009) and Fan and Liao (2011) consider covariate selection in GMM estimation; Belloni et al. (2012) and García (2011) consider selection of instruments in the GMM framework; Liao (2013) provides a shrinkage GMM method for moment selection and Cheng and Liao (2013) consider the selection of valid and relevant moments via penalized GMM; Liao and Phillips (2014) apply adaptive shrinkage techniques

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3Bai (2010, Section 6) discusses the case of multiple breaks. As he remarks, if the number of breaks is given, the one-at-a-time approach of Bai (1997b) can be used to estimate the break dates, and if the number of breaks is unknown, a test for existence of break point can be applied to each subsample before estimating a break point. Alternatively, one can use information criteria to determine the number of breaks in the latter case, but further investigation is called for.
to cointegrated systems; Kock (2013) considers Bridge estimators of static linear panel data models with random or fixed effects; Caner and Knight (2013) apply Bridge estimators to differentiate a unit root from a stationary alternative; Caner and Han (2013) proposes a Bridge estimator for pure factor models and shows the selection consistency; Lu and Su (2013) apply adaptive group Lasso to choose both regressors and the number of factors in panel data models with factor structures; Su et al. (2013) propose a procedure that is called classifier Lasso to estimate a latent panel structure; Cheng et al. (2014) provide an adaptive group Lasso estimator for pure factor structures with a one-time structural break. This paper adds to the literature by applying the shrinkage idea to panel data models with an unknown number of breaks.

We propose two approaches, penalized least squares (PLS) and penalized general method of moments (PGMM), for the estimation of the panel data model with an unknown number of breaks. We apply first differencing to remove the fixed effects in the equation and focus on the first-differenced equation. When there is no endogeneity issue in the first-differenced equation, we propose to apply the PLS to estimate the unknown number of break points and the regime-specific regression coefficients jointly where the penalty term is imposed through the adaptive group fused Lasso (AGFL) component. In the presence of endogeneity in the first-differenced equation, which may arise from endogenous regressors or lagged dependent variables in the original fixed-effects equation, we propose to apply the PGMM to estimate the unknown number of break points and the regime-specific regression coefficients jointly where, again, the penalty term is imposed through the AGFL component. Unlike Qian and Su (2013) who can only establish the claim that the group fused Lasso can not under-estimate the number of breaks in a time series regression and that all the break fractions (but not the break dates) can be consistently estimated as in Bai and Perron (1998), we show that with probability approaching one (w.p.a.1) both of our PLS and PGMM methods can correctly determine the unknown number of breaks and estimate the common break dates consistently. We obtain estimates of the regression coefficients via post Lasso and establish their asymptotic distributions. We also propose and validate a data-driven method to determine the tuning parameter used in the Lasso procedure.

Both PLS and PGMM can be numerically solved using the fast block-coordinate descent algorithm. Monte Carlo simulations show that our methods perform well in finite samples. First, the probability of correctly estimating the number of breaks (0, 1, and 2), as $N$ increases from 50 to 500, converges to 100% quickly. Even when $N = 50$ and $T = 6$, our methods are reliable in detecting the number of breaks in most cases. Second, conditional on the correct estimation of the number of breaks, our methods accurately estimate the break dates in finite samples.

As an empirical illustration, we employ our PGMM method to evaluate the effect of foreign direct investment (FDI) inflow on economic growth. We estimate a dynamic panel data model with possibly multiple breaks using the PGMM approach. We find that, with a tuning parameter selected via minimizing a BIC-type information criterion, there are four breaks (five regimes) in the span of seven five-year periods. In each regime, the post-Lasso estimation finds significant positive effect of FDI inflow on GDP.
growth. In contrast, if we estimate a usual dynamic panel data model with time-invariant parameters, we would find this effect to be negative, although statistically insignificant. This empirical example illustrates the perils of employing panel data models with restrictions on the number of breaks. Our contribution makes the restriction unnecessary.

The rest of the paper is organized as follows. Section 2 introduces our fixed-effect panel data model and PLS and PGMM estimation of the model depending on whether endogeneity is present in the first-differenced equation. Sections 3 and 4 analyze the asymptotic properties of PLS and PGMM estimators, respectively. Section 5 reports the Monte Carlo simulation results. Section 6 provides an empirical application and Section 7 concludes.

NOTATION. Throughout the paper we adopt the following notation. For an $m \times n$ real matrix $A$, we denote its transpose as $A'$, its Frobenius norm as $\|A\|_F$, and its spectral norm as $\|A\|_{sp}$. When $A$ is symmetric, we use $\mu_{\text{max}}(A)$ and $\mu_{\text{min}}(A)$ to denote its largest and smallest eigenvalues, respectively. $I_p$ denotes a $p \times p$ identity matrix and $0_{a \times b}$ an $a \times b$ matrix of zeros. We use “p.d.” to abbreviate “positive definite”. The operator $P^n$ denotes convergence in probability, $D^n$ convergence in distribution, and plim probability limit. Let $\Delta$ and $\Delta^2$ denote the difference operators of order 1 and 2, respectively. In addition, we use $\text{TriD}(\cdot, \cdot)_T$ to denote a symmetric block tridiagonal matrix:

$$\text{TriD}(A, D)_T \equiv \begin{pmatrix} D_1 & -A'_2 \\ -A_2 & D_2 & -A'_3 \\ & -A_3 & D_3 & -A'_4 \\ & & \ddots & \ddots & \ddots \\ & & & -A_{T-1} & D_{T-1} & -A'_T \\ & & & & -A_T & D_T \end{pmatrix}$$

where $D_t$'s are symmetric, $A_t$'s are square matrices, and empty blocks denote the matrices of zeros. By Molinari (2008), the determinant of $\text{TriD}(A, D)_T$ is given by $\det(\text{TriD}(A, D)_T) = \prod_{t=1}^{T} \det(A_t)$, where $A_1 = D_1$ and $A_t = D_t - A_tA_{t-1}^{-1}A_t'$ for $t = 2, \ldots, T$. By Meurant (1992) and Ran and Huang (2006), one can also calculate the inverse of $\text{TriD}(A, D)_T$ recursively.

2 Shrinkage estimation of linear panel data models with multiple breaks

In this section we consider a linear panel data model with an unknown number of breaks, which we estimate via the adaptive group fused Lasso.
2.1 The model

Consider the following linear panel data model

\[ y_{it} = \mu_i + \beta_t x_{it} + u_{it}, \quad i = 1, \ldots, N, \quad t = 1, \ldots, T \geq 2, \]  

(2.1)

where \( x_{it} \) is a \( p \times 1 \) vector of regressors, \( u_{it} \) is the error term with zero mean, \( \beta_t \) is a \( p \times 1 \) vector of unknown coefficients, and \( \mu_i \) is the individual fixed effects that may be correlated with \( x_{it} \). We assume that \( N \) passes to infinity and \( T \) can either be fixed or pass to infinity.

Like Qian and Su (2013), we assume that \( \{\beta_1, \ldots, \beta_T\} \) exhibit certain sparse nature such that the total number of distinct vectors in the set is given by \( m + 1 \), which is unknown but typically much smaller than \( T \). More specifically, we assume that

\[ \beta_t = \alpha_j \quad \text{for} \quad t = T_{j-1}, \ldots, T_j - 1 \quad \text{and} \quad j = 1, \ldots, m + 1 \]

where we adopt the convention that \( T_0 = 1 \) and \( T_{m+1} = T + 1 \). The indices \( T_1, \ldots, T_m \) indicate the unobserved \( m \) break points/dates and the number \( m + 1 \) denotes the total number of regimes. We are interested in estimating the unknown number \( m \) of unknown break dates and the regression coefficients. Let \( \alpha_m = (\alpha_1', \ldots, \alpha_m')' \) and \( T_m = \{T_1, \ldots, T_m\} \). Throughout, we denote the true value of a parameter with a superscript 0. In particular, we use \( m^0, \alpha_{m^0}' = (\alpha_{1^0}', \ldots, \alpha_{m^0+1}')' \) and \( T_{m^0} = \{T_{1^0}, \ldots, T_{m^0}\} \) to denote the true number of breaks, the vector of true regression coefficients, and the set of true break dates, respectively. We assume that \( m^0 \) is a fixed finite integer and \( T_{1^0} \geq 2 \) but allow \( T_{m^0} = T \). When \( T_{m^0} = T \), the last break occurs at the end of the sample (c.f., Andrews (2003)) and the \((m^0 + 1)\)th regime has only one observation for each individual time series.

To eliminate the effect of \( \mu_i \) in the estimation procedure, we consider the first-differenced equation

\[ \Delta y_{it} = \beta_t' x_{it} - \beta_{t-1}' x_{i,t-1} + \Delta u_{it} \]

\[ = \beta_t' \Delta x_{it} + (\beta_t - \beta_{t-1})' x_{i,t-1} + \Delta u_{it}, \]

where, e.g., \( \Delta y_{it} = y_{it} - y_{i,t-1} \) for \( i = 1, \ldots, N \) and \( t = 2, \ldots, T \). We consider two cases:

(a) \( E[\Delta u_{it} x_{it}] = 0 \) and \( E[\Delta u_{it} x_{i,t-1}] = 0 \);

(b) \( E[\Delta u_{it} x_{it}] \neq 0 \) or \( E[\Delta u_{it} x_{i,t-1}] \neq 0 \).

Case (a) occurs when \( x_{it} \) is strictly exogenous in the sense that \( E(u_{it}|x_i) = 0 \) a.s. where \( x_i = (x_{i1}, \ldots, x_{iT})' \). But strict exogeneity is not necessary for case (a) and a sufficient condition for (a) to hold is \( E(\Delta u_{it}|x_{it}, x_{it-1}) = 0 \). Case (b) occurs when \( x_{it} \) contains either lagged dependent variables (e.g., \( y_{i,t-1} \)) or endogenous regressors that are correlated with \( u_{it} \). We assume the existence of a \( q \times 1 \) vector of instruments \( z_{it} \) for \((x_{it}, x_{i,t-1})\) in case (b) where \( q \geq p \).

Since neither \( m \) nor the break dates are known and \( m \) is typically much smaller than \( T \), this motivates us to consider the estimation of \( \beta_t \)'s and \( T_m \) via a variant of fused Lasso a la Tibshirani et al. (2005). We propose two approaches – PLS estimation for case (a) and PGMM estimation for case (b).
2.2 Penalized least squares (PLS) estimation

In case (a), we propose to estimate $\beta = (\beta_1', ..., \beta_T')'$ by minimizing the following PLS objective function

$$V_{1NT, \lambda_1} (\beta) = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=2}^{T} (\Delta y_{it} - \beta_t x_{it} + \beta_{t-1} x_{i,t-1})^2 + \lambda_1 \sum_{t=2}^{T} \psi_t \left\| \beta_t - \beta_{t-1} \right\|$$

(2.2)

where $\lambda_1 = \lambda_1 (N, T) \geq 0$ is a tuning parameter, and $\psi_t$ is a data-driven weight defined by

$$\psi_t = \left\| \hat{\beta}_t - \hat{\beta}_{t-1} \right\|^{-\kappa_1}, \ t = 2, ..., T,$$

(2.3)

where $\{\hat{\beta}_t\}$ are preliminary estimates of $\{\beta_t\}$, and $\kappa_1$ is an user-specified positive constant that usually takes value 2 in the literature. Noting that the objective function in (2.2) is convex in $\beta$, it is easy to obtain the solution $\hat{\beta} = (\hat{\beta}_1', ..., \hat{\beta}_T')'$ where we suppress the dependence of $\hat{\beta}_t = \hat{\beta}_t (\lambda_1)$ on $\lambda_1$ as long as no confusion arises. We will propose a data-driven method to choose $\lambda_1$ in Section 3.4.

For a given solution $\{\hat{\beta}_t\}$ to (2.2), the set of estimated break dates are given by $\hat{T}_{\tilde{m}} = \{\hat{T}_1, ..., \hat{T}_{\tilde{m}}\}$ where $1 < \hat{T}_1 < ... < \hat{T}_{\tilde{m}} \leq T$ such that $\left\| \hat{\beta}_{j-1} - \hat{\beta}_{j-1} \right\| \neq 0$ at $t = \hat{T}_j$ for some $j \in \{1, ..., \tilde{m}\}$ and $\hat{T}_{\tilde{m}}$ divides the time interval $[1, T]$ into $\tilde{m} + 1$ regimes such that the parameter estimates remain constant within each regime. Let $\hat{T}_0 = 1$ and $\hat{T}_{\tilde{m}+1} = T + 1$. Define $\hat{\alpha}_j = \hat{\alpha}_j (\hat{T}_{\tilde{m}}) = \hat{\beta}_{\hat{T}_j-1}$ as the estimate of $\alpha_j$ for $j = 1, ..., \tilde{m} + 1$. Frequently we suppress the dependence of $\hat{\alpha}_j$ on $\hat{T}_{\tilde{m}}$ (and $\lambda_1$) unless necessary. Let $\hat{\alpha}_{\tilde{m}} = \hat{\alpha}_{\tilde{m}} (\hat{T}_{\tilde{m}}) = (\hat{\alpha}_1 (\hat{T}_{\tilde{m}})', ..., \hat{\alpha}_{\tilde{m}+1} (\hat{T}_{\tilde{m}})')'$.

Apparently, the objective function in (2.2) is closely related to the literature on the adaptive Lasso (Zou (2006)), the group Lasso (Yuan and Lin (2006)), the fused Lasso (Tibshirani et al. (2005) and Rinaldo (2009)), and group fused Lasso (Qian and Su (2013)). Like Qian and Su (2013), the use of the Frobenius norm $\| \|$ for the vector difference $\beta_t - \beta_{t-1}$ generalizes the fused Lasso to the group fused Lasso. Unlike Qian and Su (2013) who do not have any weights to use in their time series regression, our panel regression allows us to apply the adaptive weights $\{\psi_t\}$, yielding the adaptive Lasso procedure. For this reason, we can call our estimation procedure as an adaptive group fused Lasso (AGFL) procedure.

To obtain $\{\hat{\psi}_t\}$, we propose to obtain the preliminary estimate $\hat{\beta} = (\hat{\beta}_1', ..., \hat{\beta}_T')'$ by minimizing the first term in the definition of $V_{1NT, \lambda_1} (\beta)$ in (2.2). Let $\phi_{ab,ts} = \frac{1}{N} \sum_{i=1}^{N} a_i b_{it}^s$ and $\phi_{ab,t} = \phi_{ab,tt}$ for $t, s = 1, ..., T$, and $a, b = x, \Delta x, \Delta y, \Delta^2 y, \Delta u$ or $\Delta^2 u$. For example, $\phi_{x\Delta^2 y_{t,t+1}} = \frac{1}{N} \sum_{i=1}^{N} x_{it} \Delta^2 y_{i,t+1}$ for $t = 2, ..., T - 1$. We can readily demonstrate that $\hat{\beta} = \hat{Q}_{NT}^{-1} \hat{R}_{NT}$, where

$$\hat{Q}_{NT} = \text{TriD}(Q_T^y, Q_T),$$

(2.4)

$$\hat{R}_{NT} = (-\phi_{x\Delta^2 a,2}, -\phi_{x\Delta^2 a,3}, -\phi_{x\Delta^2 a,3,4}, ..., -\phi_{x\Delta^2 a,T-1,T}, \phi_{x\Delta a,T}), \ a = y \ or \ u,$$

(2.5)

$Q_t = \phi_{xx,t}$ for $t = 1$ and $T$, $Q_t = 2\phi_{xx,t}$ for $2 \leq t \leq T - 1$, and $Q_{t-1} = \phi_{xx,t-1}$ for $t = 2, ..., T$. 

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2.2.1 Post-Lasso estimation

For any $\alpha_m = (\alpha_1', ..., \alpha_{m+1}')$ and $T_m = \{T_1, ..., T_m\}$ with $1 < T_1 < \cdots < T_m \leq T$, we define\footnote{By default, the summation $\sum_{t=m}^b x_t$ is zero if $b < a$.}

$$Q_{1NT}(\alpha_m; T_m) = \frac{1}{N} \sum_{j=1}^{m+1} \sum_{t=T_{j-1}+1}^{T_j-1} \frac{1}{N} \sum_{i=1}^N (\Delta y_{it} - \alpha_j' \Delta x_{it})^2 + \frac{1}{N} \sum_{j=1}^m \sum_{i=1}^N (\Delta y_{iT_j} - \alpha_j' x_{i,T_j} + \alpha_j' x_{i,T_j-1})^2,$$

where $\sum_{t=T_{j-1}+1}^{T_j-1} \sum_{i=1}^N (\Delta y_{it} - \alpha_j' \Delta x_{it})^2$ corresponds to “the sum of squared errors” for observations in the $j$th artificial regime with time series observations indexed by integers in the interval $[T_{j-1}, T_j - 1]$, and $\sum_{i=1}^N (\Delta y_{iT_j} - \alpha_j' x_{i,T_j} + \alpha_j' x_{i,T_j-1})^2$ corresponds to the “sum of squared errors” for observations when one moves from the $j$th regime to the $(j+1)$th regime. The second term in (2.6) is important and helps to improve the asymptotic efficiency when $T$ is fixed. It can be omitted if $\min_{2 \leq j \leq m} |T_{j-1} - T_j| \to \infty$ as $N \to \infty$ and only the asymptotic efficiency is concerned, but we still keep it to improve the finite sample performance of the post-Lasso estimate in this case. One can choose $\alpha_m$ to minimize the objective function in (2.6). We denote the solution as $\tilde{\alpha}_P^m (T_m) = (\tilde{\alpha}_P^1 (T_m)', ..., \tilde{\alpha}_P^{m+1} (T_m))'$. By setting $T_m$ as $\hat{T}_m$, the set of estimated break dates via the AGFL procedure, we obtain the post-Lasso estimator

$$\tilde{\alpha}_P^m = \tilde{\alpha}_P^m (\hat{T}_m) = \hat{\Phi}^{-1}_{NT} \hat{\Psi}^p_{NT}$$

where $\hat{\Phi}_{NT}$ and $\hat{\Psi}^p_{NT}$ are $p(\tilde{m} + 1) \times p(\tilde{m} + 1)$ and $p(\tilde{m} + 1) \times 1$ matrices that are respectively defined in (A.4) and (A.5) in the appendix. We shall study the limiting distribution of $\tilde{\alpha}_P^m$ below.

2.3 Penalized GMM (PGMM) estimation

In case (b), we propose to estimate $\beta$ by minimizing the following PGMM objective function

$$V_{2NT,\lambda_2} (\beta) = \sum_{t=2}^{T} \left\{ \frac{1}{N} \sum_{i=1}^N \rho_{tt} \left( \beta_t, \beta_{t-1} \right) \right\} W_t \left\{ \frac{1}{N} \sum_{i=1}^N \rho_{tt} \left( \beta_t, \beta_{t-1} \right) \right\} + \lambda_2 \sum_{t=2}^{T} \tilde{w}_t \| \beta_t - \beta_{t-1} \|^2, \tag{2.7}$$

where $\rho_{tt} (\beta_t, \beta_{t-1}) = z_{it} (\Delta y_{it} - \beta_t x_{it} + \beta_{t-1} x_{i,t-1})$, $\lambda_2 = \lambda_2 (N, T) \geq 0$ is a tuning parameter, $W_t = W_{tNT}$ is a $q \times q$ symmetric positive definite weight matrix for $t = 2, ..., T$, and $\tilde{w}_t$ is a data-driven weight defined by

$$\tilde{w}_t = \| \beta_t - \beta_{t-1} \|^{-\kappa_2}, \ t = 2, ..., T, \tag{2.8}$$

$\{\tilde{\beta}_t\}$ are preliminary estimates of $\{\beta_t\}$, and $\kappa_2$ is an user-specified positive constant that usually takes value 2 in the literature. Clearly, the first term in the definition of $V_{2NT,\lambda_2} (\beta)$ in (2.7) is different from the usual GMM objective function in the panel setting with time-invariant parameters where only one weight matrix ($W$, say) is needed and the double summation $\sum_{t=2}^{T} \sum_{i=1}^N$ occurs twice, one before the single weight matrix and the other after the single weight matrix. It is also different from the GMM-type objective function in Andrews (1993) who considers the test of a single structural change in a time series.
regression. Noting that the objective function in (2.7) is convex in $\beta$, it is easy to obtain the solution 
$\hat{\beta} = (\hat{\beta}_1, ..., \hat{\beta}_T)'$, where we frequently suppress the dependence of $\hat{\beta}_t = \hat{\beta}_t (\lambda_2)$ on $\lambda_2$. We will propose a data-driven method to choose $\lambda_2$ in Section 4.4.

For a given solution $\{\hat{\beta}_t\}$ to (2.7), we can find the set of estimated break dates $\hat{T}_m = \{\hat{T}_1, ..., \hat{T}_m\}$ as in Section 2.2. Like before, $\hat{T}_m$ divides $[1, T]$ into $m + 1$ regimes such that the parameter estimates remain constant within each regime and $\left\| \hat{\beta}_t - \hat{\beta}_{t-1} \right\| \neq 0$ whenever $t = \hat{T}_j$ for some $j = 1, ..., \hat{m}$. Let $\hat{T}_0 = 1$ and $\hat{T}_{\hat{m}+1} = T + 1$. Define $\hat{\alpha}_j = \hat{\alpha}_j (\hat{T}_{\hat{m}}) = \hat{\beta}_{\hat{T}_{j-1}}$ as the estimate of $\alpha_j$ for $j = 1, ..., \hat{m} + 1$. Let $\hat{\alpha}_m = \hat{\alpha}_m (\hat{T}_{\hat{m}}) = (\hat{\alpha}_1 (\hat{T}_{\hat{m}})', ..., \hat{\alpha}_{\hat{m}+1} (\hat{T}_{\hat{m}})').$

To obtain the adaptive weights $\{\hat{w}_t\}$, we propose to obtain the preliminary estimate $\hat{\beta} = (\hat{\beta}_1, ..., \hat{\beta}_T)'$ by minimizing the first term in the definition of $V_{2NT, \lambda_2} (\beta)$ in (2.7). Let $\hat{Q}_{ab,t,s} = \phi_{ab,t,s} W_t \phi_{ab,t,s}$ and $\hat{Q}_{ab,t} = \hat{Q}_{ab,t,s}$ for $t, s = 1, 2, ..., T$. Let $\hat{Q}_{zx,t,t-1} = \phi_{zx,t} W_t \phi_{zx,t,t-1}$ for $t = 2, ..., T$. It is easy to show that $\hat{\beta} = \hat{Q}^{-1}_{2NT} \hat{R}_{NT}$, where

$$\hat{Q}_{2NT} = \text{TriD} \left( \hat{Q}, \hat{Q} \right)_T,$$

$$\hat{R}_{NT} = \left( - (\phi_{zx,2} W_2 \phi_{zx,t}^2), (\phi_{zx,2} W_2 \phi_{zx,t}^2 - \phi_{zx,3,2} W_3 \phi_{zx,t}^3), ..., (\phi_{zx,T-1} W_{T-1} \phi_{zx,t}^{T-1} - \phi_{zx,T,T-1} W_T \phi_{zx,t}^T), (\phi_{zx,T} W_T \phi_{zx,t}^T) \right)' ,$$

$$a = y \text{ or } u, \hat{Q}_1 = \hat{Q}_{zx,1,2}, \hat{Q}_t = \hat{Q}_{zx,t} + \hat{Q}_{zx,t+1,t} \text{ for } t = 2, ..., T - 1, \hat{Q}_T = \hat{Q}_{zx,t}, \text{ and } \hat{Q}_t = \hat{Q}_{zx,t,t-1} \text{ for } t = 2, ..., T.$$ 

### 2.3.1 Post-Lasso estimation

For any $\alpha_m = (\alpha'_1, ..., \alpha'_{m+1})'$ and $T_m = \{T_1, ..., T_m\}$ with $1 < T_1 < \cdots < T_m \leq T$, we define

$$Q_{2NT} (\alpha_m; T_m) = \sum_{j=1}^{m+1} \left[ \frac{1}{N} \sum_{t=T_{j-1}+1}^{T_j} \sum_{i=1}^{N} \rho_{it} (\alpha_j) \right]' W_j \frac{1}{N} \sum_{t=T_{j-1}+1}^{T_j} \sum_{i=1}^{N} \rho_{it} (\alpha_j) + \sum_{j=1}^{m} \left[ \frac{1}{N} \sum_{i=1}^{N} \rho_{1T_j} (\alpha_{j+1}, \alpha_j) \right]' W_{T_j} \frac{1}{N} \sum_{i=1}^{N} \rho_{1T_j} (\alpha_{j+1}, \alpha_j),$$

where $\rho_{it} (\alpha_j) = z_{it} (\Delta y_{it} - \alpha'_j \Delta x_{it}), \rho_{1T_j} (\alpha_{j+1}, \alpha_j) = z_{iT_j} (\Delta y_{iT_j} - \alpha'_{j+1} x_{iT_j} + \alpha'_{j} x_{iT_j-1})$, and $W_j$ is a regime-specific $q \times q$ symmetric weight matrix that is positive definite in large samples. As in the case of PLS estimation, the second term in (2.11) is important when $T$ is fixed and can be omitted in the case where $\min_{0 \leq j \leq m} |T_{j+1} - T_j| \to \infty$ as $N \to \infty$. Let $\hat{\alpha}_m (T_m) = (\hat{\alpha}_1 (T_m)', ..., \hat{\alpha}_{m+1} (T_m)')'$ denote the minimizer of $Q_{2NT}$ defined in (2.11). By setting $T_m$ as $\hat{T}_m$, the set of estimated break dates, we obtain the post-Lasso estimator

$$\hat{\alpha}_m = \hat{\alpha}_m (\hat{T}_m) = \hat{T}^{-1}_{NT} \hat{\xi}_{NT}.$$ 

---

5By default, the summation $\sum_{b}^{a}$ is zero if $b < a.$
where $\hat{\Sigma}_{NT}$ and $\hat{\Xi}_{NT}^p$ are $p(\hat{m} + 1) \times p(\hat{m} + 1)$ and $p(\hat{m} + 1) \times 1$ matrices that are defined in (B.3) in the appendix. We shall study the limiting distribution of $\hat{\alpha}_m^p$ below.

To obtain the PGMM estimate and the associated post-Lasso estimate, one needs to choose the weight matrices $W_t$ ($t = 2, ..., T$) and $W_j^a$ ($j = 1, ..., \hat{m} + 1$). In the simulation and application below, we adopt a two-step strategy for determining both sets of weights. For $W_t$, we first obtain the estimate $\hat{\beta}_t$ by choosing the $p \times p$ identity matrix $\mathbb{I}_p$ as the weight matrix. In the second step, we specify $W_t$ as the inverse of the estimated covariance matrix of $\rho_{it}(\hat{\beta}_t, \hat{\beta}_{t-1})$ and to achieve an updated estimate of $\beta_t$. A similar procedure is adopted for determining the weights in post-Lasso estimation.

## 3 Asymptotic properties of the PLS estimators

In this section we address the asymptotic properties of the PLS estimators.

### 3.1 Basic assumptions

Let $I_j^0 = T_j^0 - T_{j-1}^0$ for $j = 1, ..., m^0 + 1$. Define

$$I_{\min} = \min_{1 \leq j \leq m^0 + 1} I_j^0, \quad J_{\min} = \min_{1 \leq j \leq m^0} \|\alpha_{j+1}^0 - \alpha_j^0\|, \quad \text{and} \quad J_{\max} = \max_{1 \leq j \leq m^0} \|\alpha_{j+1}^0 - \alpha_j^0\|.$$  

Apparently, $I_{\min}$ denotes the minimum interval length among the $m^0 + 1$ regimes, and $J_{\min}$ and $J_{\max}$ denote the minimum and maximum jump sizes, respectively. In the case of fixed $T$, $I_{\min}$ does not pass to infinity as $N \to \infty$. If we allow $T \to \infty$, then $I_{\min}$ can either pass to infinity or stay fixed unless otherwise stated. We will maintain the assumption that $J_{\max}$ is always a fixed constant but $I_{\min}$ can be either fixed or shrinking to zero as either $N \to \infty$ or $(N, T) \to \infty$, where $(N, T) \to \infty$ denotes both $N$ and $T$ pass to infinity simultaneously.

Let $\Phi_{ab,l} = \frac{1}{N} \sum_{t = T_{l-1}^0}^{T_l^0} \sum_{s=1}^N a_{ts} b_{ts}'$ for $l = 1, ..., m^0 + 1$ and $a, b = \Delta x, x, \Delta y$ and $\Delta u$. Define the $p(\hat{m}^0 + 1) \times p(\hat{m}^0 + 1)$ matrix $\Phi_{NT}$ and $p(\hat{m}^0 + 1) \times 1$ vector $\Phi_{NT}$ and $\Psi_{NT}^q$, respectively:

$$\Phi_{NT} = \text{TriD} \left( \Phi^i, \Phi \right)_{m^0 + 1},$$

$$\Psi_{NT}^q \left( \Phi_{\Delta x \Delta a, 1} - \phi_{\Delta x \Delta a, T_{1}^0 - 1, T_{1}^0}, \Phi_{\Delta x \Delta a, 2} - \phi_{\Delta x \Delta a, T_{2}^0 - 1, T_{2}^0} + \phi_{\Delta x \Delta a, T_{1}^0}, \ldots, \Phi_{\Delta x \Delta a, m^0} - \phi_{\Delta x \Delta a, T_{m^0}^0 - 1, T_{m^0}^0} + \phi_{\Delta x \Delta a, T_{m^0 - 1}^0}, \Phi_{\Delta x \Delta a, m^0 + 1} \right), \quad a = y \text{ or } u.$$

$$\Phi_1 = \Phi_{\Delta x \Delta x, 1} + \phi_{\Delta x, T_{1}^0 - 1}, \Phi_l = \Phi_{\Delta x \Delta x, l} + \phi_{\Delta x, T_{l-1}^0} + \phi_{\Delta x, T_{l}^0 - 1} \quad \text{for} \quad l = 2, ..., m^0, \quad \Phi_{m^0 + 1} = \Phi_{\Delta x \Delta x, m^0 + 1} + \phi_{\Delta x, T_{m^0}^0}, \quad \text{and} \quad \Phi_{l+1} = \phi_{\Delta x, T_{l}^0} \quad \text{for} \quad l = 1, ..., m^0.$$  

Let $\mathbb{D}_{m^0+1} = \text{diag} \left( \sqrt{T_1^0}, \ldots, \sqrt{T_{m^0+1}^0} \right) \otimes \mathbb{I}_p$. To study the asymptotic properties of the PLS estimators, we make the following assumptions.

**Assumption A.1.** (i) Let $u_i = (u_{i1}, \ldots, u_{iT})'$. \{ $x_i, u_i$ \} are independently distributed over $i$.

(ii) $E(x_{it} \Delta u_{it}) = 0$ and $E(x_{it-1} \Delta u_{it}) = 0$ for $i = 1, \ldots, N$ and $t = 2, \ldots, T$. $\max_{1 \leq i \leq N} \max_{1 \leq t \leq T} E \| \xi_{it} \|^4 < C < \infty$ for $\xi = x$ and $u$.  

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(iii) There exists a matrix \( Q_0 > 0 \) such that \( \left\| \hat{Q}_{NT} - \hat{Q}_0 \right\|_p = O_P (1) \). There exist two constants \( \xi_0 \) and \( \xi_0' \) such that \( \xi_0 < \xi_0' \leq \lambda_{\min} (\hat{Q}_0) \leq \lambda_{\max} (\hat{Q}_0) \leq \xi_0 < \infty \).

**Assumption A.2.**

(i) \( J_{\text{max}} = O (1) \) and \( N^{1/2} J_{\text{min}} \rightarrow c, 0 < c \in (0, \infty) \) as \( N \rightarrow \infty \) or \( (N, T) \rightarrow \infty \).

(ii) \( N^{1/2} \lambda_1 J_{\text{min}}^{-1/2} \rightarrow c \in [0, \infty) \) as \( N \rightarrow \infty \) or \( (N, T) \rightarrow \infty \).

(iii) \( N^{(\kappa + 1)/2} \lambda_1 \rightarrow \infty \) as \( N \rightarrow \infty \) or \( (N, T) \rightarrow \infty \).

**Assumption A.3.**

(i) \( \mathbb{D}_{m+1}^{-1} \Psi_{NT} \mathbb{D}_{m+1}^{-1} \overset{P}{\rightarrow} \Phi_0 > 0 \).

(ii) \( \sqrt{N} \mathbb{D}_{m+1}^{-1} \Psi_{NT} \mathbb{D}_{m+1}^{-1} \overset{D}{\rightarrow} N (0, \Omega_0) \).

Assumption A.1(i) requires that \( \{x_i, u_i\} \) be independently distributed. It may be relaxed to allow for weak forms of cross sectional dependence at very lengthy arguments. A.1(ii) specifies moment conditions on \( \{x_{it}, u_{it}\} \). If \( E (u_{it} | x_{it+1}, x_{it}) = 0 \) a.s. for each \( i \) and \( t \), then the first part of A.1(i) is satisfied. In conjunction with A.1(i), A.1(ii) implies that each block element of \( \sqrt{N} \hat{R}_{NT} \) is \( O_P (1) \) and \( T^{-1} N \left\| \hat{R}_{NT} \right\|_p^2 = O_P (1) \) by Chebyshev inequality. A.1(iii) requires that the limiting matrix \( \hat{Q}_0 \) of the \( T_p \times T_p \) matrix \( \hat{Q}_{NT} \) be well behaved. Let \( \phi_{x,x,t,s} = \phi_{x,x,t,s}^0 \) and \( \phi_{x,x,t} = \phi_{x,x,t}^0 \). Let \( \Delta_t^0 = \phi_{x,x,t}^0 = 2 \phi_{x,x,t} - \phi_{x,x,t-1} (\hat{\Delta}_t^0)^{-1} \phi_{x,x,t-1} \) for \( t = 2, \ldots, T-1 \), and \( \Delta_T^0 = \phi_{x,x,T} - \phi_{x,x,T-1} (\hat{\Delta}_T^0)^{-1} \phi_{x,x,T-1} \). Then \( \hat{Q}_0 \) is p.d. if and only if the sequence of matrices \( \{\Delta_0^0, \ldots, \hat{\Delta}_T^0\} \) are all p.d. Combining A.1(i)-(iii), we prove in Lemma A.1 that \( \sqrt{N} \left( \hat{\beta}_t - \beta_t^0 \right) = O_P (1) \) for each \( t = 1, \ldots, T \). Assumption A.2(i) mainly specifies conditions on \( J_{\text{min}}, \lambda_1, \) and \( N \). Note that we allow the minimum break size \( J_{\text{min}} \) to shrink to zero as \( N \rightarrow \infty \) but it cannot shrink to zero faster than \( N^{-1/2} \). In the special case where \( J_{\text{min}} \) is bounded away from zero, A.2 can be simplified to

**Assumption A.2'.** \( N^{1/2} \lambda_1 \rightarrow c \in [0, \infty) \) and \( N^{(\kappa + 1)/2} \lambda_1 \rightarrow \infty \) as \( N \rightarrow \infty \) or \( (N, T) \rightarrow \infty \).

Assumption A.3 specify conditions to ensure the asymptotic normality of the post Lasso estimators.

### 3.2 Consistency

The following theorem establishes the consistency of \( \{\hat{\beta}_t\} \).

**Theorem 3.1** Suppose that Assumption A.1 holds. Then (i) \( T^{-1} \left\| \hat{\beta} - \beta^0 \right\|_p^2 = O_P \left( N^{-1} \right) \), and (ii) \( \hat{\beta}_t - \beta_t^0 = O_P \left( N^{-1/2} \right) \) for each \( t = 1, \ldots, T \).

Theorem 3.1(i) and (ii) establish the mean square and pointwise convergence rates of \( \{\hat{\beta}_t\} \), respectively. The two results are equivalent in the case of fixed \( T \). When \( T \) is allowed to pass to infinity as \( N \rightarrow \infty \), the proof of Theorem 3.1(ii) demands some extra effort. In particular, we need a close examination of the factorization and inversion properties of symmetric block tridiagonal matrices. See the proof of Theorem 3.1(ii) in Appendix A.

Let \( T_{m^2}^0 = \{2, \ldots, T\} \setminus T_{m^2}^0 \). Let \( \theta_t^0 = \beta_t^0 \) and \( \theta_t^0 = \beta_t^0 - \beta_{t-1}^0 \) for \( t = 2, \ldots, T \). Let \( \hat{\theta}_1 = \hat{\beta}_1 \) and \( \hat{\theta}_t = \hat{\beta}_t - \hat{\beta}_{t-1} \) for \( t = 2, \ldots, T \). The following theorem establishes the selection consistency.
Theorem 3.2 Suppose that Assumptions A.1-A.2 hold. Then \( P\left( \| \hat{\theta}_t \| = 0 \text{ for all } t \in T_m^0 \right) \to 1 \) as \( N \to \infty \).

Theorem 3.2 says that w.p.a.1 all the zero vectors in \( \{ \theta_t^0, 2 \leq t \leq T \} \) must be estimated as exactly zero by the PLS method so that the number of estimated breaks \( \hat{m} \) cannot be larger than \( m^0 \) when \( N \) is sufficiently large. On the other hand, by Theorem 3.1(ii), we know that the estimates of the nonzero vectors in \( \{ \theta_t^0, 2 \leq t \leq T \} \) must be consistent by noting that \( \hat{\beta}_t - \tilde{\beta}_{t-1} \) consistently estimates \( \theta_t^0 \) for \( t \geq 2 \).

Put together, Theorems 3.1 and 3.2 imply that the AGFL has the ability to identify the true regression dates) can be consistently estimated as in Bai and Perron (1998). More precisely, letting \( N_T \) be the number of structural changes and all the break dates consistently regardless of whether \( T \) is fixed or passes to infinity. In contrast, Qian and Su (2013, Theorem 3.3) only establish the claim that the group fused Lasso procedure can not under-estimate the number of breaks in a time series regression and that all the break fractions (but not the break dates) can be consistently estimated as in Bai and Perron (1998).

Corollary 3.3 Suppose that Assumptions A.1-A.2 hold with \( c_J = \infty \) in Assumption A.2(i). Then (i) \( \lim_{N \to \infty} P \left( \hat{m} = m^0 \right) = 1 \), and (ii) \( \lim_{N \to \infty} P(\tilde{T}_1 = T_1^0, \ldots, \tilde{T}_m^0 = T_m^0 \mid \hat{m} = m^0) = 1 \).

The above corollary implies that, as long as \( J_{\min} \) remains fixed or shrinks to zero at a rate slower than \( N^{-1/2} \) as \( N \to \infty \), we can estimate the number of structural changes and all the break dates consistently regardless of whether \( T \) is fixed or passes to infinity. In our panel setting, the availability of \( N \) cross sectional units for each time period permits us to obtain the set of consistent preliminary estimates \( \{ \hat{\beta}_t \} \) used to construct the adaptive weights \( \{ \hat{w}_t \} \).

The adaptive nature of our group fused Lasso procedure helps us to identify the exact set of break dates and yields stronger results than those in Qian and Su (2013).

3.3 Limiting distribution of the post-Lasso estimator

In this subsection we study the asymptotic distribution of the post-Lasso estimator \( \tilde{\alpha}_m^p(\tilde{T}_m) \). Corollary 3.3 implies that w.p.a.1, \( \hat{m} = m^0 \) and \( \tilde{T}_j = T_j^0 \) for \( j = 1, \ldots, m^0 \). It follows that \( \tilde{\alpha}_m^p(\tilde{T}_m) \) is asymptotically equivalent to the infeasible estimator \( \tilde{\alpha}_m^p(T_m^0) \) which is obtained if one knows the exact set \( T_m^0 \) of true break dates. Note that

\[
\tilde{\alpha}_m^p(T_m^0) = \Phi_{NT}^{-1} \Psi_{NT}^p
\]

where \( \Phi_{NT} \) and \( \Psi_{NT}^p \) are defined in (3.1) and (3.2), respectively.

The following theorem reports the limiting distribution of \( \tilde{\alpha}_m^p(\tilde{T}_m) \) conditional on the large probability event \( \{ \hat{m} = m^0 \} \).

Theorem 3.4 Suppose that Assumptions A.1-A.3 hold with \( c_J = \infty \) in Assumption A.2(i). Then conditional on \( \hat{m} = m^0 \), we have

\[
\sqrt{N} \tilde{\alpha}_m^p(\tilde{T}_m) - \alpha(0) \overset{D}{\to} N \left( 0, \Phi_0^{-1} \Omega_0 \Phi_0^{-1} \right).
\]
Since we allow $I^0_j$ to be either fixed or diverge to infinity in the case of large $T$, $\hat{\alpha}^0_{j}(\hat{T}_m)$’s may have different convergence rates to their true values. In the special case where $I^0_j$ is proportional to $T$, $\hat{\alpha}^0_{j}(\hat{T}_m)$ achieves the usual $\sqrt{NT}$-rate of consistency.

### 3.4 Choosing the tuning parameter $\lambda_1$

Let $\tilde{\alpha}_{m_{\lambda_1}} \equiv \tilde{\alpha}_{m_{\lambda_1}}(\hat{T}_{m_{\lambda_1}}) = (\tilde{\alpha}_1(\hat{T}_{m_{\lambda_1}}), ..., \tilde{\alpha}_{m_{\lambda_1}+1}(\hat{T}_{m_{\lambda_1}}))'$ denote the set of post-Lasso estimates of the regression coefficients based on the break dates in $\hat{T}_{m_{\lambda_1}} = \hat{T}_{m_{\lambda_1}}(\lambda_1)$, where we make the dependence of various estimates on $\lambda_1$ explicit. Let $\hat{\sigma}^2_{\hat{m}_{\lambda_1}} \equiv \frac{1}{NT} Q_{1NT}(\tilde{\alpha}_{m_{\lambda_1}}; \hat{T}_{m_{\lambda_1}})$. We propose to select the tuning parameter $\lambda_1$ by minimizing the following information criterion:

$$IC(\lambda_1) = \frac{\hat{\sigma}^2_{\hat{m}_{\lambda_1}}}{\min \lambda_1} + \rho_{1NT} \cdot (\min \lambda_1 + 1).$$  \hfill (3.3)

Denote $\Omega = [0, \lambda_{\max}]$, a bounded interval in $\mathbb{R}^+$. We divide $\Omega$ into three subsets $\Omega_0$, $\Omega_-$ and $\Omega_+$ as follows

$$\Omega_0 = \{ \lambda_1 \in \Omega : \min \lambda_1 = m^0 \}, \quad \Omega_- = \{ \lambda_1 \in \Omega : \min \lambda_1 < m^0 \}, \quad \text{and} \quad \Omega_+ = \{ \lambda_1 \in \Omega : \min \lambda_1 > m^0 \}.$$

Clearly, $\Omega_0$, $\Omega_-$ and $\Omega_+$ denote the three subsets of $\Omega$ in which the correct-, under- and over-number of breaks are selected by the adaptive group fused Lasso, respectively. Let $\lambda^0_{0NT}$ denote an element in $\Omega_0$ that satisfies the conditions on $\lambda_1$ in Assumptions A.2(ii)-(iii).

Let $\hat{\sigma}^2_{N_T} = \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} (\Delta u_{it})^2$ and $\sigma^2_0 \equiv \text{plim} \hat{\sigma}^2_{N_T}$. To state the next result, we add the following assumptions.

**Assumption A.4.** (i) $\text{plim}_{N \to \infty} \text{min}_{1 \leq j \leq m_0} \text{min}_{\alpha \in \mathbb{R}^p} \frac{1}{N_{j_{min}}} \sum_{i=1}^{N} [(\alpha^0_{j+1} - \alpha) x_{it} T^0_j - (\alpha^0_j - \alpha) x_{i, T^0_j - 1}]^2 \geq c_\alpha > 0$.

(ii) $\frac{1}{N(T-1)} \sum_{t=2}^{T} \sum_{i=1}^{N} \Delta x_{it} \Delta u_{it} = O_P(1)$.

(iii) As $N \to \infty$ or $(N, T) \to \infty$, $\frac{T}{(\text{min}_{j_{\min}})^2} \to 0$.

**Assumption A.5.** As $N \to \infty$ or $(N, T) \to \infty$, $\left(1 + \frac{T}{(\text{min}_{j_{\min}})^2} \right) \rho_{1NT} \to 0$ and $N \rho_{1NT} \to \infty$.

A.4(i) imposes conditions on the parameters and the observations that are either at the break dates or immediately preceding the break dates. The scalar $J^2_{\text{min}}$ reflects the fact that we allow the minimum break size $J_{\text{min}}$ to shrink to zero. In the latter case, pulling observations in two adjacent regimes with the break size of order $O(J_{\text{min}})$ together to estimate the regression coefficients within these two regimes is still consistent with $J_{\text{min}}^{-1}$-rate of consistency. Under A.2(i)-(ii), A.4(ii) can be verified under various weak dependence conditions, say, strong mixing or martingale difference sequence-type of conditions. A.4(iii) imposes restriction on $I_{\text{min}}$, $J_{\text{min}}$ and the sample sizes. It is trivially satisfied if $I_{\text{min}} \propto T$ and $J_{\text{min}}$ remains fixed as $N \to \infty$ or $(N, T) \to \infty$, and reduces to the condition that $c_j = \infty$ in Assumption A.2(i) in the case where $T$ is fixed. A.5 reflects the usual conditions for the consistency of model selection, that is, the penalty coefficient $\rho_{1NT}$ cannot shrink to zero either too fast or too slowly. If $I_{\text{min}} \propto T$ and $J_{\text{min}}^{-1} = O(1)$,
the first part of A.5 requires that \( \rho_{1,NT} \to 0 \), which is standard for an information-criterion function. \( N^{-1} \) indicates the probability order of the distance between the first term in the criterion function for an over-parametrized model and that for the true model.

**Theorem 3.5** Suppose that Assumptions A.1, A.2(i) and A.3-A.5 hold with \( c_J = \infty \) in Assumption A.2(i). Then

\[
P \left( \inf_{\lambda_1 \in \Omega_1 \cup \Omega_2} \text{IC} (\lambda_1) > \text{IC} (\lambda_{1,NT}^0) \right) \to 1 \quad \text{as} \quad N \to \infty.
\]

Theorem 3.5 implies that the \( \lambda_1 \)'s that yield the over-estimated or under-estimated number of breaks fail to minimize the information criterion w.p.a.1. Consequently, the minimizer of \( \text{IC} (\lambda_1) \) can only be the one that produces the correct number of estimated breaks in large samples. Note that we prove the above theorem without requiring \( \lambda_1 \) to satisfy Assumptions A.2(ii)-(iii). It indicates that if the number of corrected breaks is of our major concern, we can simply choose \( \lambda_1 \) to minimize \( \text{IC} (\lambda_1) \).

### 4 Asymptotic properties of the PGMM estimators

In this section we address the statistical properties of the PGMM estimators.

#### 4.1 Assumptions

Let \( \phi_{ab,l+1} = \phi_{ab,T_l^0} W_{T_l^0} \phi_{ab,T_l^0} T_l^0 \) for \( l = 1,\ldots,m^0 \) and \( a,b = z, x, \Delta x \). Define the \( p(m^0 + 1) \times p(m^0 + 1) \) matrix \( \Upsilon_{NT} \) and \( \mathbf{P} (m^0 + 1) \times 1 \) vector \( \Xi_{NT} \), respectively:

\[
\Upsilon_{NT} = \text{TriD} (\Upsilon_{m^0+1},\Upsilon_{m^0+1}^\top,\Upsilon_{m^0+1}^\top,\Upsilon_{m^0+1}^\top,\Upsilon_{m^0+1}^\top), \quad \Xi_{NT} = (\Xi_{a,1},\Xi_{a,2},\ldots,\Xi_{a,m^0+1})^\top, \quad a = y \text{ or } u, \tag{4.1}
\]

where \( \Upsilon_1 = \Phi'_{\Delta x,l+1} W_{T_l^0} \Phi_{\Delta x,l+1} + \phi_{xx,l+1} W_{T_l^0} \phi_{xx,l+1} T_{l-1}^0 - W_{T_l^0} \phi_{xx,l+1} T_{l-1}^0 - W_{T_l^0} \phi_{xx,l+1} T_{l-1}^0 
+ \phi_{xx,l+1} W_{T_l^0} \phi_{xx,l+1} T_{l-1}^0 \) for \( l = 2,\ldots,m^0 \), \( \Upsilon_{m^0+1} = \Phi'_{\Delta x,m^0+1} W_{m^0+1} \Phi_{\Delta x,m^0+1} + \phi_{xx,m^0+1} W_{m^0+1} \phi_{xx,m^0+1} T_{m^0}^0 \phi_{xx,m^0+1} \), and \( \Upsilon_l^0 = \phi_{xx,l} W_{T_l^0} \phi_{xx,l} T_{l-1}^0 \) for \( l = 2,\ldots,m^0+1 \). In addition, for \( a = y \text{ or } u, \Xi_{a,1} = \Phi'_{\Delta x,l} W_{m^0} \Phi_{\Delta x,l} + \phi_{xx,l} W_{m^0} \phi_{xx,l} T_{m^0}^0 \phi_{xx,l} T_{m^0}^0 \), \( \Xi_{a,l} = \Phi'_{\Delta x,l} W_{m^0} \Phi_{\Delta x,l} T_{m^0}^0 - \phi_{xx,l} W_{m^0} \phi_{xx,l} T_{m^0}^0 - \phi_{xx,l} W_{m^0} \phi_{xx,l} T_{m^0}^0 \phi_{xx,l} T_{m^0}^0 \) for \( l = 2,\ldots,m^0 \), and \( \Xi_{a,m^0+1} = \Phi'_{\Delta x,m^0+1} W_{m^0+1} \Phi_{\Delta x,m^0+1} + \phi_{xx,m^0+1} W_{m^0+1} \phi_{xx,m^0+1} T_{m^0}^0 \).

To study the asymptotic properties of the PGMM estimators, we make the following assumptions.

**Assumption B.1.** (i) Let \( z_i = (z_{i1},\ldots,z_{iT})^\top \). \( \{x_i, z_i, u_i\} \) are independently distributed over \( i \).

(ii) \( E (z_{it} u_{it}) = 0 \) for \( i = 1,\ldots,N \) and \( 2 = 1,\ldots,T \). \( \max_{1 \leq i \leq N} \max_{1 \leq t \leq T} E \|z_{it}\|^4 < C \) for \( \zeta_{it} = x_{it}, z_{it}, \) and \( u_{it} \).

(iii) There exists a matrix \( \hat{Q}_0 > 0 \) such that \( \|Q_{NT} - \hat{Q}_0\|_p = o_P (1) \). There exist two constants \( C_{0,0} \) and \( \tilde{C}_{0,0} \) such that \( 0 < C_{0,0} < \lambda_{\min}(\hat{Q}_0) \leq \lambda_{\max}(\hat{Q}_0) \leq \tilde{C}_{0,0} < \infty \).

**Assumption B.2.** (i) \( J_{\max} = O (1) \) and \( N^{1/2} J_{\min} \to c_J \in (0, \infty] \) as \( N \to \infty \) or \( (N,T) \to \infty \).

(ii) \( N^{1/2} \lambda_2 J_{\min}^{\varepsilon_2} \to c \in [0, \infty) \) as \( N \to \infty \) or \( (N,T) \to \infty \).
(iii) $N^{(\kappa_2+1)/2}\lambda_2 \to \infty$ as $N \to \infty$ or $(N, T) \to \infty$.

Assumption B.3. (i) $\mathbb{D}_m^{-1} \Gamma_{NT} \mathbb{D}_m^{-1} \overset{P}{\rightarrow} \mathbb{Y}_0 > 0$.
(ii) $\sqrt{N} \mathbb{D}_m^{-1} \mathbb{E}_N \mathbb{D}_m^{-1} \overset{D}{\rightarrow} N(0, \Sigma_0)$.

Assumptions B.1(i)-(iii) parallel Assumptions A.1(i)-(iii). B.1(ii) specifies moment conditions on $\{x_{it}, z_{it}, u_{it}\}$. In conjunction with B.1(i), B.1(ii) implies that each block element of $\sqrt{N} \hat{R}_{NT}$ is $O_P(1)$ and $T^{-1} N \left\| \hat{R}_{NT} \right\|^2 = O_P(1)$ by Chebyshev inequality. Combining B.1(i)-(iii), we prove in Lemma B.1 that $\sqrt{N} (\hat{\beta}_t - \beta_t^0) = O_P(1)$ for each $t = 1, ..., T$. Assumption B.2(i) mainly specifies conditions on $J_{\min}$, $\lambda_2$, and $N$. Note that we allow the minimum break size $J_{\min}$ to shrink to zero as $N \to \infty$. In the special case where $J_{\min}$ is bounded away from zero, B.2 can be simplified to

**Assumption B.2**. $N^{1/2}\lambda_2 \rightarrow c \in [0, \infty)$ and $N^{(\kappa_2+1)/2}\lambda_2 \rightarrow \infty$ as $N \to \infty$ or $(N, T) \to \infty$.

Assumption B.3 specify conditions to ensure the asymptotic normality of the post Lasso estimator.

### 4.2 Consistency

The following theorem establishes the consistency of $\{\hat{\beta}_t\}$.

**Theorem 4.1** Suppose that Assumption B.1 holds. Then (i) $T^{-1} \left\| \hat{\beta} - \beta^0 \right\|^2 = O_P \left( N^{-1} \right)$, and (ii) $\hat{\beta}_t - \beta_t^0 = O_P \left( N^{-1/2} \right)$ for each $t = 1, ..., T$.

Theorems 4.1(i) and (ii) establish the mean square and pointwise convergence rates of $\{\hat{\beta}_t\}$, respectively. The two results are equivalent in the case of fixed $T$ and are not in the case of large $T$. If $T \to \infty$ as $N \to \infty$, the proof of Theorem 4.1(ii) requires the use of the factorization and inversion properties of symmetric block tridiagonal matrices as in the proof of Theorem 3.1(ii).

Let $\hat{\theta}_t = \hat{\beta}_t$ and $\hat{\theta}_t = \hat{\beta}_t - \hat{\beta}_{t-1}$ for $t = 2, ..., T$. The following theorem establishes the selection consistency.

**Theorem 4.2** Suppose that Assumptions B.1-B.2 hold. Then $P \left( \left\| \hat{\theta}_t \right\| = 0 \text{ for all } t \in T_{m_0}^{00} \right) \to 1$ as $N \to \infty$.

Theorem 4.2 says that w.p.a.1 all the zero vectors in $\{\theta_t^0, 2 \leq t \leq T\}$ must be estimated as exactly zero by the PGMM method. On the other hand, by Theorem 4.1(ii), we know that the estimates of the nonzero vectors in $\{\theta_t^0, 2 \leq t \leq T\}$ must be consistent by noting that $\hat{\beta}_t - \hat{\beta}_{t-1}$ consistently estimates $\theta_t^0$ for $t \geq 2$. Put together, Theorems 4.1 and 4.2 imply that the AGFL has the ability to identify the true regression model with the correct number of breaks consistently when the minimum break size $J_{\min}$ does not shrink to zero too fast.

**Corollary 4.3** Suppose that Assumptions B.1-B.2 hold with $c_J = \infty$ in Assumption B.2(i). Then (i) $\lim_{N \to \infty} P (\hat{m} = m^0) = 1$ and (ii) $\lim_{N \to \infty} P (\hat{T}_1 = T_1^0, ..., \hat{T}_{m_0} = T_{m_0}^0 | \hat{m} = m^0) = 1$. 

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The above corollary implies that the PGMM method helps us to estimate the number of structural changes and all the break dates consistently regardless of whether \( T \) is fixed or passes to infinity.

### 4.3 Limiting distribution of the post-Lasso estimator

In this subsection we study the asymptotic distribution of the post-Lasso estimator \( \hat{\alpha}_m^P(\hat{T}_m) \). Corollary 4.3 implies that w.p.a.1, \( \hat{m} = m^0 \) and \( \hat{T}_j = T^0_j \) for \( j = 1, \ldots, m^0 \). It follows that \( \hat{\alpha}_m^P(\hat{T}_m) \) is asymptotically equivalent to the infeasible estimator \( \hat{\alpha}_m^P(T^0_m) \) which is obtained if one knows the exact set \( T^0_m \) of true break dates. Note that

\[
\hat{\alpha}_m^P(T^0_m) = \hat{\gamma}_N^{-1} N_T^T
\]

where \( \hat{\gamma}_N \) and \( N_T^T \) are defined in (4.1).

The following theorem reports the limiting distribution of \( \hat{\alpha}_m^P(\hat{T}_m) \) conditional on the large probability event \( \{ \hat{m} = m^0 \} \).

**Theorem 4.4** Suppose that Assumptions B.1-B.3 hold. Then conditional on \( \hat{m} = m^0 \), we have \( \sqrt{N} \{ \hat{\alpha}_m^P(\hat{T}_m) - \alpha_0 \} \overset{d}{\rightarrow} N \left( 0, \Sigma_{N_0}^{-1} \Sigma_0^{-1} \right) \).

Since we allow \( T^0_j \) to be either fixed or diverge to infinity in the case of large \( T \), \( \hat{\alpha}_m^P(\hat{T}_m) \)’s may have different convergence rates to their true values. In the special case where \( T^0_j \) is proportional to \( T \), \( \hat{\alpha}_m^P(\hat{T}_m) \) achieves the usual \( \sqrt{NT} \)-rate of consistency.

### 4.4 Choosing the tuning parameter \( \lambda_2 \)

Let \( \hat{\alpha}_m^{\lambda_{2+}} = \hat{\alpha}_m^{\lambda_{2-}}(\hat{T}_m) = (\hat{\alpha}_1(\hat{T}_m), \ldots, \hat{\alpha}_m^{\lambda_{2+}}(\hat{T}_m))' \) denote the set of post-Lasso estimates of the regression coefficients based on the break dates in \( \hat{T}_m^{\lambda_{2+}} = \hat{T}_m^{\lambda_{2-}}(\lambda_2) \), where we make the dependence of various estimates on \( \lambda_2 \) explicit. Let \( \hat{\sigma}^2_{\hat{T}_m^{\lambda_{2+}}} = \frac{1}{m} \sum_{i=1}^m \hat{\sigma}^2_{\hat{T}_m^{\lambda_{2+}}} \). We propose to select the tuning parameter \( \lambda_2 \) by minimizing the following information criterion:

\[
IC_2(\lambda_2) = \hat{\sigma}^2_{\hat{T}_m^{\lambda_{2+}}} + \rho_{\lambda_2} \hat{T}_m^{\lambda_{2+}} (\hat{m}_2 + 1). \tag{4.2}
\]

Denote \( \Omega_2 = [0, \lambda_{2_{\text{max}}}^0] \), a bounded interval in \( \mathbb{R}^+ \). We divide \( \Omega_2 \) into three subsets \( \Omega_{20}, \Omega_{2-} \) and \( \Omega_{2+} \) as follows

\[
\Omega_{20} = \{ \lambda_2 \in \Omega_2 : \hat{m}_2 = m^0 \}, \quad \Omega_{2-} = \{ \lambda_2 \in \Omega_2 : \hat{m}_2 < m^0 \}, \quad \text{and} \quad \Omega_{2+} = \{ \lambda_2 \in \Omega_2 : \hat{m}_2 > m^0 \}.
\]

Let \( \lambda_{2_{\text{opt}}}^0 \) denote an element in \( \Omega_{20} \) that also satisfies the conditions on \( \lambda_2 \) in Assumptions B.2(ii)-(iii).

To state the next result, we add the following assumptions.

**Assumption B.4.**

(i) \( \lim_{N \rightarrow \infty} \min_{1 \leq j \leq m^0} \min_{\alpha \in \mathbb{R}} \frac{1}{T_{\min}} \eta_j(\alpha)' W_{T^0_j, \eta_j}(\alpha) \geq c_0 \geq 0 \),

(ii) \( \frac{1}{\sqrt{N} \min_{j+1}} \sum_{i=s_j+1}^{T_{j+1}^0-1} \sum_{i=s_{j+1}}^{s_j} z_i \Delta u_i = O_p(1) \) for each \( j = 1, \ldots, m^0 + 1 \).
Assumption B.5. As $N \to \infty$ or $(N, T) \to \infty$, \( \left(1 + \frac{T}{\min J_{\min}}\right) \rho_{2 NT} \to 0 \) and $N \rho_{2 NT} \to \infty$.

Assumptions B.4-B.5 parallel A.4-A.5. Note that we now require $\min J \to \infty$ in the case of large $T$. The following theorem implies that the minimizer of $IC_2(\lambda_2)$ can only be the one that produces the correct number of estimated breaks in large samples.

Theorem 4.5 Suppose that Assumptions B.1, B.2(i) and B.3-B.5 hold with $c_J = \infty$ in Assumption B.2(i). Then

\[
P \left( \inf_{\lambda_2 \in \Omega_{2-} \cup \Omega_{2+}} IC_2(\lambda_2) > IC_2(\lambda^0_{2 NT}) \right) \to 1 \quad \text{as} \quad N \to \infty.
\]

5 Monte Carlo simulations

In this section we conduct a set of Monte Carlo experiments to evaluate the finite sample performance of our AGFL method. The first set of experiments are concerned with the PLS or PGMM estimation of static panel data models. We first evaluate the probability of falsely detecting breaks when there are none. Then we experiment on the data generating processes (DGP) with one or two breaks. In this case, we evaluate both the probability of correctly detecting the number of breaks and the accuracy of estimating the break dates. The second set of experiments deal with the PGMM estimation of dynamic panel data models. We focus on DGP with a lagged dependent variable and an exogenous variable. Like in the static panel case, we evaluate the probability of correctly detecting the number of breaks and, when there are indeed breaks, the accuracy of break date estimation.

For fast computation, we use the block-coordinate descent algorithm (see, e.g., Angelosante and Giannakis (2012)) to solve the minimization problem in (2.2) for the PLS case and (2.7) for the PGMM case. We select the tuning parameters $\lambda_1$ and $\lambda_2$ that minimize the information criterion in (3.3) and (4.2) for the cases of PLS and PGMM estimation, respectively. Specifically, we choose a tuning parameter $\lambda^{\max}$ that would yield zero break in every DGP and a $\lambda^{\min}$ that would yield many breaks. In practice, we can easily find such $\lambda^{\max}$ and $\lambda^{\min}$ by trial and error. We then search for the optimal tuning parameter on the 40 evenly-distributed logarithmic grids in the interval $[\lambda^{\min}, \lambda^{\max}]$. We choose $\rho_{1 NT} = \rho_{2 NT} = 1/\sqrt{N(T-1)}$ in (3.3) or (4.2) for the static panel and $\rho_{2 NT} = \log(N(T-1))/(N(T-1))$ in (4.2) for the dynamic panel. Note that the latter choice specifies exactly the same rate as required by the Bayesian Information Criterion (BIC). Both choices are acceptable in theory for every DGP we experiment on, but their finite-sample performances do differ.

Following the literature on adaptive Lasso, we set $\kappa_1 = \kappa_2 = 2$ in the construction of the adaptive weights $\{\hat{w}_i\}$ and $\{\hat{w}_j\}$ that are used for the PLS and PGMM estimation, respectively. In addition, we choose all weight matrices $\{W_t, t = 2, ..., T\}$ and $\{W^p_j, j = 1, ..., \hat{m} + 1\}$ as detailed in the last paragraph of Section 2.3. The number of repetitions in all subsequent Monte Carlo experiments is 500.
5.1 The case of static panel

We consider the following DGPs:

\[ y_{it} = \beta_t x_{it} + \mu_t + u_{it}, \quad i = 1, \ldots, N, \quad t = 1, \ldots, T, \]

where \( \mu_t = T^{-1} \sum_{t=1}^{T} x_{it} \) and

- **DGP-1**: \( x_{it} \sim \text{i.i.d. } N(0, 1) \), \( u_{it} = \sigma_u \eta_{it}, \eta_{it} \sim \text{i.i.d. } N(0, 1) \).
- **DGP-2**: Same as DGP-1 except \( \eta_{it} \sim \text{AR}(1) \) for each \( i : \eta_{it} = 0.5 \eta_{i,t-1} + \epsilon_{it}, \epsilon_{it} \sim \text{i.i.d. } N(0, 0.75) \).
- **DGP-3**: Same as DGP-1 except \( \eta_{it} \sim \text{GARCH}(1, 1) \) for each \( i : \eta_{it} = \sqrt{h_{it}} \epsilon_{it}, h_{it} = 0.05 + 0.05 \eta_{i,t-1}^2 + 0.9 h_{i,t-1}, \epsilon_{it} \sim \text{i.i.d. } N(0, 1) \).
- **DGP-4**: \( x_{it} = \xi_{it} + 0.3 \eta_{it}, \eta_{it} \text{ and } \xi_{it} \text{ are i.i.d. } N(0, 1) \) and mutually independent, \( u_{it} = \sigma_u \eta_{it}, z_{it} = \xi_{it} + 0.3 \epsilon_{it}, \epsilon_{it} \sim \text{i.i.d. } N(0, 1) \).
- **DGP-5**: Same as DGP-4 except \( \xi_{it} \sim \text{AR}(1) \) for each \( i : \xi_{it} = 0.5 \xi_{i,t-1} + \epsilon_{it}, \epsilon_{it} \sim \text{i.i.d. } N(0, 0.75) \).
- **DGP-6**: Same as DGP-4 except \( \eta_{it} \sim \text{GARCH}(1, 1) \) for each \( i : \eta_{it} = \sqrt{h_{it}} \epsilon_{it}, h_{it} = 0.05 + 0.05 \eta_{i,t-1}^2 + 0.9 h_{i,t-1}, \epsilon_{it} \sim \text{i.i.d. } N(0, 1) \).

We consider \( T = 6 \) or 12, and \( N = 50, 100, 200, \) and 500. For each DGP, we set \( \beta_t = 1 \) for all \( t \) when no break exists, \( \beta_t = 1 \{1 \leq t \leq T/2\} \) when there is one break, and \( \beta_t = 1 \{1 \leq t \leq T/2\} + 1 \{T/2 < t \leq 2T/3\} \) when there are two breaks, where \( 1 \{\cdot\} \) denotes the usual indicator function. If \( T = 6 \), the last case allows consecutive breaks at \( t = 4 \) and 5.

Note that the individual effects are generated from within-average and thus regarded as “fixed effects”. In the first three DGPs, no endogeneity issue exists and we use PLS to estimate the models. DGP-1 serves as the benchmark case where both the regressor and the idiosyncratic error processes are strong white noise. DGP-2 allows serial correlation in the idiosyncratic error process and DGP-3 allows conditional heteroskedasticity. The DGP-4 through 6 contain an endogenous variable \( x_{it} \) and a variable \( z_{it} \) that generates a valid IV. We apply PGMM to estimate the models, using \((z_{it}, z_{i,t-1})\)' as the instrument. DGP-4 serves as the benchmark case where both the regressor and the error terms are i.i.d. across \( i \) and \( t \). \( x_{it} \) and \( u_{it} \) are correlated due to the common component \( \eta_{it} \), and \( z_{it} \) is correlated with \( x_{it} \) due to the presence of \( \xi_{it} \) in both. DGP-5 allows serial correlation in \( x_{it} \), and DGP-6 allows conditional heteroskedasticity in \( u_{it} \).

To evaluate the performance of the PLS or PGMM estimation under different noise levels, we select the scale parameter \( \sigma_u \) to be \( \sqrt{1/2} \) and 1. In DGP-1, these values for \( \sigma_u \) correspond to signal-to-noise ratios of 2 and 1 (or in terms of the goodness of fit \( R^2 \) of the model, 0.67 and 0.5), respectively.

Tables 1 and 2 report simulation results from the above DGPs. The first panel of Table 1 reports the percentages of falsely detecting breaks when there are none \((m^0 = 0)\). The second and the third panels
Table 1: The determination of the number of breaks for DGPs 1-6 (static panels)

<table>
<thead>
<tr>
<th>DGP</th>
<th>$\sigma_u$</th>
<th>$N = 50$</th>
<th>$N = 100$</th>
<th>$N = 200$</th>
<th>$N = 500$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$T = 6$</td>
<td>$T = 12$</td>
<td>$T = 6$</td>
<td>$T = 12$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$m^0 = 0$, % of falsely detecting breaks when there are none.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$\frac{\sqrt{2}}{1}$</td>
<td>0.2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{\sqrt{2}}{1}$</td>
<td>5.6</td>
<td>1.8</td>
<td>1.6</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{\sqrt{2}}{1}$</td>
<td>0.2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{\sqrt{2}}{1}$</td>
<td>6.8</td>
<td>3.6</td>
<td>1</td>
<td>0.6</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{\sqrt{2}}{1}$</td>
<td>0.2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{\sqrt{2}}{1}$</td>
<td>0.4</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

$\sigma_u = 1$, % of correctly detecting one break

<table>
<thead>
<tr>
<th>DGP</th>
<th>$\sigma_u$</th>
<th>$N = 50$</th>
<th>$N = 100$</th>
<th>$N = 200$</th>
<th>$N = 500$</th>
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<tbody>
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<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$\frac{\sqrt{2}}{1}$</td>
<td>99.4</td>
<td>99.8</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{\sqrt{2}}{1}$</td>
<td>94.4</td>
<td>95.8</td>
<td>99.0</td>
<td>99.8</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{\sqrt{2}}{1}$</td>
<td>99.8</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{\sqrt{2}}{1}$</td>
<td>95.8</td>
<td>95.8</td>
<td>99.8</td>
<td>99.6</td>
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<tr>
<td>5</td>
<td>$\frac{\sqrt{2}}{1}$</td>
<td>85.4</td>
<td>79.2</td>
<td>97.2</td>
<td>96.4</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{\sqrt{2}}{1}$</td>
<td>87.6</td>
<td>95.2</td>
<td>95.4</td>
<td>100</td>
</tr>
</tbody>
</table>

$\sigma_u = 2$, % of correctly detecting two breaks

<table>
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<tr>
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<th>$N = 100$</th>
<th>$N = 200$</th>
<th>$N = 500$</th>
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<td>$T = 12$</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$\frac{\sqrt{2}}{1}$</td>
<td>99.0</td>
<td>99.8</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
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<td>$\frac{\sqrt{2}}{1}$</td>
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<td>97.4</td>
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<td>98.8</td>
</tr>
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<td>$\frac{\sqrt{2}}{1}$</td>
<td>99.8</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{\sqrt{2}}{1}$</td>
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<td>99.8</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
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<td>$\frac{\sqrt{2}}{1}$</td>
<td>93.2</td>
<td>96.2</td>
<td>99.0</td>
<td>99.6</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{\sqrt{2}}{1}$</td>
<td>12.2</td>
<td>43.0</td>
<td>27.8</td>
<td>77.8</td>
</tr>
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19
Table 2: The accuracy of estimating the break dates for DGPs 1-6 (static panels)

<table>
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<th>DGP</th>
<th>$\sigma_{\mu}$</th>
<th>$N = 50$</th>
<th>$N = 100$</th>
<th>$N = 200$</th>
<th>$N = 500$</th>
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<tbody>
<tr>
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<td>$T = 6$</td>
<td>12</td>
<td>6</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
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<td>$\chi^2_1$</td>
<td>.034</td>
<td>.017</td>
<td>.000</td>
<td>.000</td>
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<tr>
<td></td>
<td>$\chi^2_1$</td>
<td>.000</td>
<td>.035</td>
<td>.000</td>
<td>.000</td>
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<tr>
<td>2</td>
<td>$\chi^2_1$</td>
<td>.000</td>
<td>.000</td>
<td>.000</td>
<td>.000</td>
</tr>
<tr>
<td></td>
<td>$\chi^2_1$</td>
<td>.000</td>
<td>.000</td>
<td>.000</td>
<td>.000</td>
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<tr>
<td>3</td>
<td>$\chi^2_1$</td>
<td>.000</td>
<td>.000</td>
<td>.000</td>
<td>.000</td>
</tr>
<tr>
<td></td>
<td>$\chi^2_1$</td>
<td>.000</td>
<td>.000</td>
<td>.000</td>
<td>.000</td>
</tr>
<tr>
<td>4</td>
<td>$\chi^2_1$</td>
<td>.070</td>
<td>.070</td>
<td>.000</td>
<td>.000</td>
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<tr>
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<td>$\chi^2_1$</td>
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<td>.463</td>
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<td>.018</td>
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<td>$\chi^2_1$</td>
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<td>.035</td>
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<tr>
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<td>$\chi^2_1$</td>
<td>1.240</td>
<td>2.037</td>
<td>.532</td>
<td>.616</td>
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</table>

Note: The table reports the ratio of the average Hausdorff distance between the estimated and true sets of break dates to $T$, i.e., $100 \cdot \text{HDD}(\hat{T}_{m,0^0}, T_{m,0})/T$ in DGPs 1-3 and $100 \cdot \text{HDD}(\hat{T}_{m,0^0}, T_{m,0})/T$ in DGPs 4-6.
report the percentages of correctly estimating the number of breaks when the true number of breaks is 1 and 2, respectively. We summarize some important findings from Table 1. First, when there are no breaks, the probability of falsely detecting breaks declines to zero as either \(N\) or \(T\) increases. This is true for both the PLS estimation in DGP-1 to DGP-3 in the case of no endogenous regressor and the PGMM estimation in DGP-4 to DGP-6 in the case of an endogenous regressor. Even with \(N = 50\) and \(T = 6\), the probabilities of false detection of breaks are very small for all DGPs under investigation. Second, when there is one break, the probabilities of correctly detecting one break converge to 100% as \(N\) increases. In the case of PLS, the probabilities of correct detection are high at both noise levels even when \(N = 50\) and \(T = 6\). In the case of PGMM, however, they are much lower at the high noise level than at the low noise level when sample sizes are small, although they converge quickly to 100% as \(N\) increases. As \(T\) increases from 6 to 12, the probability of correct detection improves in general. Third, when there are two breaks, the probabilities of correctly detecting two breaks converge to 1 as \(N\) increases from 50 to 500. When \(T = 6\), there are two consecutive breaks at \(t = 4\) and 5 and the percentage of correctly estimating the number of breaks tends to very low in DGP-4 to DGP-6 if \(N\) is not large enough (50 or 100). But it improves quickly when \(T\) increases to 12, in which case there are no consecutive breaks. It also tends to 1 rapidly as \(N\) increases from 50 to 500.

Table 2 reports the ratio of average Hausdorff distance (HD) between the estimated and true sets of break dates, i.e., 100 \(\frac{\text{HD}((\hat{T}_0^m, T_{m0}^0)/T}{\text{HD}((T_0^m, T_{m0}^0)/T}\) in the case of PLS estimation and 100 \(\frac{\text{HD}((\hat{T}_0^m, T_{m0}^0)/T}{\text{HD}((T_0^m, T_{m0}^0)/T}\) in the case of PGMM estimation, conditional on correct estimation of the number of breaks.\(^6\) Conditional on the correct estimation of the number of breaks, both PLS and PGMM estimate the break dates very accurately. Even with \(N = 50\) and \(T = 6\), the average ratios of the Hausdorff distance to the true set of breaks are close to zero for PLS at both noise levels. For DGPs with endogeneity, the estimation of break-dates is only slightly less accurate.

### 5.2 The case of dynamic panel

We consider the following DGPs with an AR(1) dynamics:

\[
y_{it} = \beta_{1t}y_{i,t-1} + \beta_{2t}x_{2it} + \mu_i + u_{it},
\]

where \(\mu_i \sim \text{i.i.d. Uniform}[-0.1, 0.1]\) and

- DGP-1d: \(x_{2it} \sim \text{i.i.d. } \mathcal{N}(0, 1), u_{it} = \sigma_u \eta_{it}, \eta_{it} \sim \text{i.i.d. } \mathcal{N}(0, 1)\).
- DGP-2d: Same as DGP-1d except \(x_{2it} \sim \text{AR}(1)\) for each \(i : x_{2it} = 0.5x_{2i,t-1} + v_{it}, v_{it} \sim \text{i.i.d. } \mathcal{N}(0, 0.75)\).

\(^6\)Let \(\mathcal{D}(A, B) \equiv \sup_{a \in B} \inf_{b \in A} |a - b|\) for any two sets \(A\) and \(B\). The Hausdorff distance between \(A\) and \(B\) is defined as

\(\text{HD}(A, B) \equiv \max \{\mathcal{D}(A, B), \mathcal{D}(B, A)\}\).
• DGP-3d: Same as DGP-1d except \( \eta_{it} \sim \text{GARCH}(1,1) \) for each \( i : \eta_{it} = \sqrt{h_{it}} \epsilon_{it}, h_{it} = 0.05 + 0.05 \eta_{i,t-1}^2 + 0.9 h_{i,t-1}, \epsilon_{it} \sim i.i.d. N(0,1) \).

As in the static case, we take \( T = 6 \) or 12, and \( N = 50, 100, 200, \) and 500. For each DGP, we set either \( \beta_{1t} = \beta_{2t} = 0.5 \) or more persistently, \( \beta_{1t} = \beta_{2t} = 0.8 \) for all \( t \) when no break exists, \( \beta_{1t} = \beta_{2t} = 0.3 \cdot \{1 \leq t \leq T/2\} + 0.7 \cdot \{T/2 < t \leq T\} \) when there is one break, and \( \beta_{1t} = \beta_{2t} = 0.3 \cdot 1 \{1 \leq t \leq T/2\} + 0.7 \cdot 1 \{T/2 + 1 \leq t < 2T/3\} + 0.3 \cdot 1 \{2T/3 + 1 \leq t \leq T\} \) when there are two breaks. Note that when \( T = 6 \), there are consecutive breaks at \( t = 4 \) and 5.

DGP-1d is the benchmark case with i.i.d. \( x_{it} \) and \( u_{it} \) across both \( i \) and \( t \). DGP-2d allows serial correlation in \( x_{it} \) and DGP-3d allows conditional heteroskedasticity in \( u_{it} \). We choose the scale parameter \( \sigma_u \) to be 0.2, 0.3, and 0.5, corresponding to signal-to-noise ratio 4, 2, and 1, respectively, in DGP-1d with \( \beta_{1t} = \beta_{2t} = 0.5 \). The relatively lower noise levels are justified by the usually high goodness-of-fit of many dynamic panels in applications. To obtain the PGMM estimate, we use \( z_{it} = (y_{i,t-2}, x_{2i,t}, x_{2i,t-1}) \) as the instrument.

Table 3 reports the estimation of the number of breaks for these three DGPs. The first two panels report the percentages of falsely detecting breaks when there are none \( (m = 0) \). The AR coefficient is 0.5 in the first panel and 0.8 in the second panel. The second and the third panels report the percentages of correctly estimating the number of breaks when the true number of breaks is 1 and 2, respectively. We summarize the results in Table 3. (i) When there are no breaks, the probabilities of falsely detecting breaks are small and become smaller in general when \( N \) or \( T \) increases. When the AR coefficient increases from 0.5 to 0.8 and the dynamic panel becomes more persistent, the probabilities of false detection remain low. In fact, for some DGPs (e.g., DGP-3d, \( N = 500 \)), the probabilities of false detection at higher persistency level are generally lower than those at moderate persistency level, thanks to the fact that the signal-to-noise ratio is higher at high persistence level. (ii) When there is one break, the probabilities of correctly detecting one break converge to 100% across all choices of \( N, T, \) and noise levels. (iii) When there are two breaks, we see relatively lower probabilities of correct estimation, especially at high noise levels. But as \( N \) increases, the probabilities of correction estimation also converge to 100% across all noise levels.

Table 4 reports the ratio of average Hausdorff distance between the estimated and true sets of break dates to \( T \), i.e., \( 100 \cdot \frac{\text{HD}(\hat{T}_{m0}^{\theta}, T_{m0}^{\theta})}{T} \) for DGP-1d to DGP 3-d. As in the static panel case, conditional on the correct estimation of the number of breaks, our procedure estimates the break dates very accurately. Even with \( N = 50 \) and \( T = 6 \), the average Hausdorff distance to the true set of break dates is very close to zero at all noise levels.
Table 3: The determination of the number of breaks for DGPs 1d-3d (dynamic panels)

<table>
<thead>
<tr>
<th>DGP</th>
<th>( N = 50 )</th>
<th>( N = 100 )</th>
<th>( N = 200 )</th>
<th>( N = 500 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \sigma_u )</td>
<td>( T = 6 )</td>
<td>( T = 12 )</td>
<td>( T = 6 )</td>
</tr>
<tr>
<td>( m^u = 0 ), ( \beta_{1u} = \beta_{2u} = 0.5 ), % of falsely detecting breaks when there are none.</td>
<td>( m^u = 0 ), ( \beta_{1u} = \beta_{2u} = 0.8 ), % of falsely detecting breaks when there are none.</td>
<td>( m^u = 1 ), % of correctly detecting one break</td>
<td>( m^u = 2 ), % of correctly detecting two breaks</td>
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<tr>
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<td>1.6</td>
<td>.6</td>
</tr>
<tr>
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<td>.5</td>
<td>2.8</td>
<td>1.2</td>
<td>1.6</td>
</tr>
<tr>
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<td>1.4</td>
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Table 4: The accuracy of estimating the break dates for DGPs 1d-3d (dynamic panels)

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<th>DGP</th>
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<th>$N = 200$</th>
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<td></td>
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<tr>
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<td>$\nu = 1$</td>
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<td>.386</td>
<td>.606</td>
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</table>

| $m^\nu = 2$ |            | $\nu = 2$ | $\nu = 2$ | $\nu = 2$ | $\nu = 2$ |
| 1d  | .2         | .000     | .000     | .000     | .000     |
|     | .3         | .000     | .041     | .079     | .000     |
|     | .5         | .000     | .572     | .273     | .000     |
| 2d  | .2         | .000     | .000     | .000     | .000     |
|     | .3         | .000     | .000     | .137     | .000     |
|     | .5         | .000     | .184     | .174     | .000     |
| 3d  | .2         | .000     | .000     | .000     | .000     |
|     | .3         | .000     | .000     | .000     | .000     |
|     | .5         | .000     | .000     | .000     | .000     |

Note: The table reports the ratio of the average Hausdorff distance between the estimated and true sets of break dates to $T$, i.e., $100 \cdot \text{HD}(\hat{T}_m^0, T_m^0)/T$.

6 An empirical application

In this section we offer an illustration of the use of our method. We seek to evaluate the effect of FDI inflow on economic growth with a dynamic panel data model with an unknown number of breaks. The possible existence of breaks may be justified theoretically. In the endogenous growth model of Romer (1986), for example, economic growth may behave differently in different policy environments. Furthermore, in the growth model of Jones (2002), the regime shifts may be common across countries in “a world of ideas”, assuming that ideas propagate fast enough. Empirically, there is ample evidence of the existence of breaks in growth path (e.g., Ben-David and Papell (1995)). However, most of existing studies rely on time series structural break tests for individual economies, the United States in particular.

In this empirical exercise, we use a panel data of 88 countries or regions from 1973 to 2012. We take data from the UNCTAD (United Nations Conference on Trade and Development) and construct two variables, the per capita GDP growth and the ratio of FDI inflow to GDP for each economy in the sample.\footnote{The UNCTAD database covers 237 countries and regions. We delete those economies with missing values over 1973-2012.} These are annual data. But following the literature on growth empirics (e.g., Islam (1995)), we take five-year averages of the two variables and denote them by $y_{it}$ and $fdi_{it}$, respectively. Here the subscript $t$ denotes a sequence of five-year periods. The averaging gives us eight time five-year time periods for each economy. Due to the fact that there is one lagged dependent variable in the model, the...
effective number of data points for each economy is seven. We apply the PGMM method to estimate the following dynamic panel data model with an unknown number of breaks,

\[ y_{it} = \mu_i + \beta_{1} y_{i,t-1} + \beta_{2} f d_{i,t} + u_{it}, \quad t = 1, \ldots, 7. \]

As in the simulations, we set \(\kappa_2 = 2\) in the construction of the adaptive weights, choose the weight matrices \((W_t, W_j^p)\) as detailed in the last paragraph of Section 2.3, and adopt \(z_{it} = (y_{i,t-2}, f d_{i,t}, f d_{i,t-1})'\) as the instrument.

We first select an optimal tuning parameter that minimizes the BIC by choosing \(\rho_{2NT} = \log(N(T - 1))/(N(T - 1))\) in (4.2). We choose \(\lambda^{\text{max}} = 10\), which results in zero break, and \(\lambda^{\text{min}} = 0.01\), which results in six breaks. We then search on the interval \([\lambda^{\text{min}}, \lambda^{\text{max}}]\) with thirty evenly-distributed logarithmic grids. We find that the number of breaks is four and that the breaks occur at \(t = 2, 5, 6, 7\), that is 1983-1987, 1998-2002, 2003-2007, and 2008-2012. Figure 1 shows how BIC (left axis) and the estimated number of breaks (right axis) change with the tuning parameter \(\lambda_2\). We can see that the BIC declines till the estimated number of breaks reaches four and rises as \(\lambda_2\) gets bigger. It is notable that there are five \(\lambda_2\)'s that result in four breaks, ranging from 0.053 to 0.137, and the IC curve is flat over this segment (and
similarly over several other segments).\(^8\) This suggests that the penalized GMM estimation is not very sensitive to the tuning parameter.

It is well known that BIC, or other information criteria, may not be able to select the right model in finite samples. It is thus prudent to examine the cases with the number of breaks other than four. Table 5 shows regime segmentation, parameter estimates, and standard errors (in parentheses), from the post-lasso estimation for the cases where \(\hat{m} = 0, 1, \ldots, 6\). Note that in the last case (\(\hat{m} = 6\)), there is a structural break at every time point.

As shown in Table 5, the set of break dates is an increasing sequence as the tuning parameter decreases. It starts from an empty set when \(\hat{m} = 0\). When \(\hat{m} = 1\), we have one break at \(t = 2\), which corresponds to the five-year period of 1983-1987. As the tuning parameter decreases, another break (in addition to the one at \(t = 2\)) is detected at \(t = 7\), which corresponds to 2008-2012. When \(\hat{m} = 3\), we have an additional break at \(t = 6\), corresponding to 2003-2007. As the tuning parameter decreases more, we arrive at the case of \(\hat{m} = 4\) that achieves the minimum BIC. When \(\hat{m} = 5\), there is another break at \(t = 3\) and the set of break dates becomes \{2, 3, 5, 6, 7\}.

Table 5 also shows that the determination of structural change in our model is crucial for the quantitative evaluation of the effect of FDI on the economic growth. If we assume that no break exists and estimate a textbook dynamic panel data model, then we may conclude that FDI has a negative, albeit statistically insignificant, effect on growth. In the model chosen by BIC (\(\hat{m} = 4\)), in stark contrast, the coefficient of FDI is significantly positive in all regimes. In models with more than four break dates, there are also negative coefficients on FDI in the five-year span of 1983-1987. In models with less than four but more than or equal to one breaks, the coefficients on FDI are positive in all regimes, but are not statistically significant in some regimes. This exercise suggests that the time-invariant parameter in the textbook dynamic panel data model is an unnecessarily restrictive assumption and may lead to dubious conclusions. Our shrinkage-based method, by allowing multiple breaks in panel data model, provides applied economists with a natural approach to relaxing this assumption.

### 7 Conclusion

We propose two shrinkage procedures for the determination of the number of structural changes in linear panel data models via adaptive group fused Lasso: PLS estimation for first-differenced models without endogeneity and PGMM estimation for first-differenced models with endogeneity. We show that with probability tending to one our methods can correctly determine the number of breaks and estimate the break dates consistently. Simulation results suggest that our methods perform well in finite samples.

There are several interesting topics for further research. First, we do not allow cross sectional de-

\(^8\)When \(\lambda_2\) changes from 0.053 to 0.137, the number of breaks and the set of estimated break dates remain unchanged so that neither the first term (corresponding to the post Lasso regression) nor the second term (the penalty term) in (4.2) changes.
Table 5: The effect of FDI on the economic growth (88 countries and regions, 1973-2012)

<table>
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<tr>
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<td>$fdi_{i,t}$</td>
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<tr>
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<tr>
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<td>$y_{i,t-1}$</td>
<td>$fdi_{i,t}$</td>
<td>-.127(.097)</td>
<td>.654(.275)**</td>
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<td>4</td>
<td>$y_{i,t-1}$</td>
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<td>-.114(.103)</td>
<td>.617(.294)**</td>
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<td>5</td>
<td>$y_{i,t-1}$</td>
<td>$fdi_{i,t}$</td>
<td>-.109(.115)</td>
<td>.625(.287)**</td>
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<td>6</td>
<td>$y_{i,t-1}$</td>
<td>$fdi_{i,t}$</td>
<td>-.107(.117)</td>
<td>.653(.285)**</td>
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Note: Numbers in parentheses are standard errors. ** denotes statistical significance at 1% level, * at 5% level, and * at 10% level.
dependence in our models. Given the large literature on cross sectional dependence, it is interesting to extend our methodology to panel data models with cross sectional dependence. Second, if we model the cross sectional dependence through a factor structure, the factor loadings may also exhibit structural changes over time (see, e.g., Breitung and Eickmeier (2011) and Cheng et al. (2014)) and this further complicates the analysis. Third, we consider the common shocks for homogenous panel data models. It is also interesting to consider heterogeneous panel data models and to allow the break dates to be different across individuals. We leave these topics for future research.
A Proof of the results in Section 3

Let $V_{1NT} (\beta) = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=2}^{T} (\Delta y_{it} - \beta'_t x_{it} + \beta'_{t-1} x_{i,t-1})^2$. Let $\beta_t = \beta'_t + N^{-1/2} b_t$ for $t = 1, ..., T$ with $b \equiv (b'_1, ..., b'_T)'$ satisfying that $T^{-1/2} \|b\| = L$. Note that $\beta = \beta'_0 + N^{-1/2} b$. We first prove a technical lemma.

**Lemma A.1** Suppose Assumption A.1 holds. Then $\beta_t - \beta'_t = O_P \left( N^{-1/2} \right)$ for each $t = 1, 2, ..., T$.

**Proof.** Let $b_t = N^{1/2} (\beta_t - \beta'_t)$ and $\hat{b} = (\hat{b}'_1, ..., \hat{b}'_T)'$. Noting that $\Delta y_{it} - x'_t \beta_t + x'_{i,t-1} \beta_{t-1} = \Delta u_{it} - N^{-1/2} (x'_t b_t - x'_{i,t-1} b_{t-1})$, we have

$$N \left[ V_{1NT} (\beta) - V_{1NT} (\beta') \right] = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=2}^{T} \left\{ \left[ \Delta u_{it} - N^{-1/2} (x'_t b_t - x'_{i,t-1} b_{t-1}) \right] - (\Delta u_{it})^2 \right\}$$

$$= \frac{1}{N} \sum_{i=1}^{N} \sum_{t=2}^{T} (x'_t b_t - x'_{i,t-1} b_{t-1})^2 - \frac{2}{N^{1/2}} \sum_{i=1}^{N} \sum_{t=2}^{T} \Delta u_{it} (x'_t b_t - x'_{i,t-1} b_{t-1})$$

$$= b' \hat{Q}_{NT} b - 2 b' \sqrt{N} \hat{R}_{NT} = A_1 (b) - 2 A_2 (b), \text{ say},$$

where $\hat{Q}_{NT}$ and $\hat{R}_{NT}$ are defined in (2.4) and (2.5), respectively. Under Assumption A.1(iii), w.p.a.1

$$\lambda_{\min} (\hat{Q}_{NT}) = \min_{\|x\|=1} \left\{ x' \hat{Q}_{0} x + x' (\hat{Q}_{NT} - \hat{Q}_{0}) x \right\} \geq \lambda_{\min} (\hat{Q}_{0}) - \left\| \hat{Q}_{NT} - \hat{Q}_{0} \right\|_{sp} \geq c \hat{q}/2.$$

Under Assumptions A.1(i)-(ii), $T^{-1/2} \|\sqrt{N} \hat{R}_{NT}\| = O_P (1)$ by Chebyshev inequality. It follows that w.p.a.1

$$T^{-1/2} \|b\| = L \text{ is sufficiently large in which case the quadratic term } A_1 (b) \text{ dominates the linear term } A_2 (b) \text{. Consequently, } N \left[ V_{1NT} (\beta) - V_{1NT} (\beta') \right] > 0 \text{ w.p.a.1 if } T^{-1/2} \|b\| = L \text{ is large and } V_{1NT} (\beta) \text{ cannot be minimized in this case. This further implies that } T^{-1/2} \|\hat{b}\| \text{ must be stochastically bounded.}

When $T$ is fixed, the above result also implies that $\hat{b}$ is stochastically bounded for each $t = 1, ..., T$. We now consider the case of large $T$. Let $\hat{L}$ denote the block lower part of the symmetric block tridiagonal matrix $\hat{Q}_{NT}$. By Meunant (1995), $\hat{Q}_{NT}$ can be factorized as follows: $\hat{Q}_{NT} = (\hat{\Delta} + \hat{L}) \hat{\Delta}^{-1} (\hat{\Delta} + \hat{L})'$, where $\hat{\Delta} = \text{diag}(\hat{\Delta}_1, ..., \hat{\Delta}_T)$ is a block diagonal matrix, $\hat{\Delta}_1 = \phi_{xx,1}, \hat{\Delta}_t = 2 \phi_{xx,t} - \phi_{xx,t,t-1} (\hat{\Delta}_{t-1})^{-1} \phi'_{xx,t,t-1}$ for $t = 2, ..., T - 1$, and $\hat{\Delta}_T = \phi_{xx,T} - \phi_{xx,T,T-1} (\hat{\Delta}_{T-1})^{-1} \phi'_{xx,T,T-1}$. Let $b^t = (\Delta + \hat{L}) b = (b'_1, ..., b'_T)'$ and $\hat{R}_{nt} = \sqrt{N} (\Delta + \hat{L}) \hat{R}_{nt} = (R'_{t1}, ..., R'_{tT})'$ where $b^t$’s and $\hat{R}_{nt}$’s are all $p \times 1$ vectors. In addition, $\hat{R}_{nt} = O_P (1)$ for each $t = 1, ..., T$ under Assumption A.1. Then

$$N \left[ V_{1NT} (\beta) - V_{1NT} (\beta') \right] = \sum_{t=1}^{T} \left[ b^t' \hat{\Delta}^{-1} b^t + b^t' \hat{R}_{nt} \right] \equiv V_{1NT} (b^t), \text{ say.}$$
Let $\beta \equiv (\beta_1', \ldots, \beta_T')'$ and $\hat{\beta} \equiv (\Delta + L') \hat{\beta} \equiv (\hat{\beta}_1', \ldots, \hat{\beta}_T')'$. In view of the fact that $\hat{\beta}$ minimizes $V_{1NT}(\beta)$, we have

$$0 \geq N \left[ V_{1NT}(\hat{\beta}) - V_{1NT}(\beta^0) \right] = \sum_{t=1}^{T} \left[ \hat{b}_t^T \Delta_t^{-1} \hat{b}_t + \hat{b}_t^T R_{t,NT}^{(1)} \right].$$

The last result implies that $\hat{b}_t = O_P(1)$ for each $t$. Otherwise, $\hat{b}_t$ cannot minimize $V_{1NT}(b_t)$, which further implies that $\hat{\beta}$ cannot minimize $V_{1NT}(\beta)$.

To finish the proof, we still need to show that $\hat{b}_t = O_P(1)$ for each $t$ based on the result that $\hat{b}_t = O_P(1)$ for each $t$. Noting that $\Delta + L'$ is a nonsingular upper block triangular matrix, we can apply the fact that the inverse of a nonsingular upper block triangular matrix is also an upper block triangular matrix (see, e.g., Harville (1997, p.95)) and write $(\Delta + L')^{-1} = \{\omega_{ts}\}_{t,s=1}^{T}$, where $\omega_{ts}$'s are $p \times p$ matrices that are $O_P(1)$ for $s \geq t$ and zeros otherwise. The exact formula of $\omega_{ts}$ in terms of elements in $\Delta$ and $L'$ can be obtained recursively from Harville (1997, p.95). Thus $\hat{b}_t = \sum_{s=1}^{T} \omega_{ts} \hat{b}_s$ and $\hat{b}_t = O_P(1)$ for any $t \geq T - r$ where $r$ is a finite integer that does not depend on $T$. Now, suppose that $\hat{b}_t \neq O_P(1)$ for some $1 \leq \tau < T - r$. By the relationship between $\hat{b}_t$ and $\hat{b}_t$, we have

$$\hat{b}_t \tau = \Delta \hat{b}_t + \phi_{x,x,t+1,\tau} \hat{b}_{t+1},$$

or, equivalently,

$$\Delta^{-1} \hat{b}_t = \hat{b}_\tau + \Delta^{-1} \phi_{x,x,t+1,\tau} \hat{b}_{t+1}.$$

Since $\Delta^{-1} = O_P(1)$, $\hat{b}_t = O_P(1)$, $\phi_{x,x,t+1,\tau} = O_P(1)$, and $\hat{b}_t \neq O_P(1)$, in order for the above equality to hold, we must have $\hat{b}_t \neq O_P(1)$. Deducting recursively, we must have $\hat{b}_{T-t} \neq O_P(1)$, a contradiction. It follows that $\hat{b}_t = N^{1/2}(\hat{\beta}_t - \beta_t^0) = O_P(1)$ for each $t$.

**Proof of Theorem 3.1.** (i) Let $\hat{b}_t = N^{1/2}(\hat{\beta}_t - \beta_t^0)$ and $\hat{\beta} = N^{1/2}(\hat{\beta} - \beta^0)$. Noting that $\Delta u_{it} = x_{it}^T \hat{\beta}_t + x_{i,t-1} \beta_t - x_{i,t-1} \beta_{t-1}$, we have

$$N \left[ V_{1NT, \lambda}(\beta) - V_{1NT, \lambda}(\beta^0) \right] = \frac{1}{N} \sum_{i=1}^{N} \left( \sum_{t=2}^{T} \left( x_{it}^T \hat{b}_t - x_{i,t-1} \beta_{t-1} \right)^2 \right) - \frac{2}{N} \sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{s=1}^{T} \Delta u_{it} \left( x_{it}^T \hat{b}_t - x_{i,t-1} \beta_{t-1} \right)$$

$$+ N \lambda_1 \sum_{t=2}^{T} \hat{w}_t \left[ \left\| \beta_t^0 - \beta_{t-1}^0 + N^{-1/2} (b_t - b_{t-1}) \right\| - \left\| \beta_t^0 - \beta_{t-1}^0 \right\| \right]$$

$$= \mathbf{b}^T Q_{1NT} \mathbf{b} - 2 \mathbf{b}^T \sqrt{N} \hat{R}_{NT}^{(1)} + N \lambda_1 \sum_{t=2}^{T} \hat{w}_t \left[ \left\| \beta_t^0 - \beta_{t-1}^0 + N^{-1/2} (b_t - b_{t-1}) \right\| - \left\| \beta_t^0 - \beta_{t-1}^0 \right\| \right]$$

$$+ N \lambda_1 \sum_{t=2}^{T} \hat{w}_t \left[ N^{-1/2} (b_t - b_{t-1}) \right]$$

$$\equiv A_1(\mathbf{b}) - 2A_2(\mathbf{b}) + A_3(\mathbf{b}) + A_4(\mathbf{b}),$$

say,
where \( \hat{Q}_{NT} \) and \( \hat{R}_{NT}^f \) are defined in (2.4) and (2.5), respectively. By Lemma A.1 and Assumption A.2(i), 
\[
\max_{t \in T_{m_0}^0} \hat{\psi}_t = \max_{s \in T_{m_0}^0} \left\| \beta_t - \beta_{t-1} \right\| > \min_{t \in T_{m_0}^0} \left\| \beta_t - \beta_{t-1} + O_P \left( N^{-1/2} \right) \right\|^2 = O_P \left( J_{\min}^{-\kappa_1} \right).
\]

By the triangle and Jensen inequalities, the fact that \( m_0 \) is a fixed finite integer, and Assumption A.2(ii),
\[
\left| T^{-1} A_3 (b) \right| \leq m_0 T^{-1} N^{1/2} \lambda_1 \max_{s \in T_{m_0}^0} \hat{\psi}_s \left\{ \frac{1}{m_0} \sum_{t=2, t \in T_{m_0}^0} \left\| b_t - b_{t-1} \right\| \right\}^{1/2} 
\]
\[
\leq m_0 T^{-1} N^{1/2} \lambda_1 \max_{s \in T_{m_0}^0} \hat{\psi}_s \left\{ \frac{1}{m_0} \sum_{t=2, t \in T_{m_0}^0} \left\| b_t - b_{t-1} \right\| \right\}^{1/2} 
\]
\[
\leq (2m_0)^{1/2} T^{-1/2} N^{1/2} \lambda_1 \max_{s \in T_{m_0}^0} \hat{\psi}_s T^{-1/2} \left\| b \right\| 
= O_P \left( \lambda_1 T^{-1/2} J_{\min}^{-\kappa_1} \right) T^{-1/2} \left\| b \right\| = O_P \left( 1 \right) T^{-1/2} \left\| b \right\|. \quad (A.1)
\]
In conjunction with the analyses of \( A_1 (b) \) and \( A_2 (b) \) in the proof of Lemma A.1, this implies that w.p.a.1
\[
T^{-1} [A_1 (b) - 2A_2 (b) + A_3 (b)] \geq \min \left( \hat{Q}_{NT} \right) T^{-1} \left\| b \right\|^2 - O_P (1) T^{-1/2} \left\| b \right\| > 0
\]
if \( T^{-1/2} \left\| b \right\| = L \) is sufficiently large. That is, \( A_1 (b) \) dominates \(-2A_2 (b) + A_3 (b)\) for large \( L \). In addition, \( A_4 (b) \geq 0 \). Consequently, \( N \left[ V_{1NT, \lambda_1} (\beta) - V_{1NT, \lambda_1} (\beta^0) \right] > 0 \) w.p.a.1 for large \( L \) and \( V_{1NT, \lambda_1} (\beta) \) cannot be minimized in this case. This further implies that \( T^{-1/2} \left\| \hat{b} \right\| \) has to be stochastically bounded and Theorem 3.1 (i) holds.

(ii) The result follows from (i) in the case of fixed \( T \). So we consider the case of large \( T \). Let \( \hat{L}, \hat{\Delta}, \hat{b}^f, \hat{R}_{NT}^f, \) and \( \{ \omega_{ts} \}_{t, s = 1}^T \) be as defined in the proof of Lemma A.1. Let \( \omega_{ts}^f = \omega_{ts} - \omega_{t-1, s} \). Then \( b_t - b_{t-1} = \sum_{s=1}^T \omega_{ts} b_t^s - \sum_{s=1}^{T-1} \omega_{t-1, s} b_t^s = \sum_{s=1}^{T-1} \omega_{ts}^f b_t^s \) as \( \omega_{ts} = 0 \) for \( s = t - 1 \). So we can rewrite \( N \left[ V_{1NT, \lambda_1} (\beta) - V_{1NT, \lambda_1} (\beta^0) \right] \) in terms of \( \hat{b}^f \):
\[
N \left[ V_{1NT, \lambda_1} (\beta) - V_{1NT, \lambda_1} (\beta^0) \right] = \sum_{t=1}^T \left[ b_t^f \hat{\Delta}_t^{-1} b_t^f - 2 b_t^f \hat{R}_{nt}^f \right] + N \lambda_1 \sum_{t=2, t \in T_{m_0}^0} \hat{\psi}_t \left\| \beta_t - \beta_{t-1} + N^{-1/2} \sum_{s=t-1}^T \omega_{ts}^f b_t^s \right\| - \left\| \beta_t - \beta_{t-1} \right\| 
+ N^{1/2} \lambda_1 \sum_{t=2, t \in T_{m_0}^0} \hat{\psi}_t \left\| \sum_{s=t-1}^T \omega_{ts}^f b_t^s \right\| \equiv NV_{1NT, \lambda_1}^f (b^f), \quad \text{say.}
\]

Noting that \( \left\| \hat{R}_{nt}^f \right\| = O_P (1), N^{1/2} \lambda_1 m_0 \max_{s \in T_{m_0}^0} \hat{\psi}_s = O_P \left( N^{1/2} \lambda_1 J_{\min}^{-\kappa_1} \right) = O_P (1), \) and \( \max_{s \in T_{m_0}^0} \left\| \omega_{ts}^f \right\| = O_P (1) \), we have by the triangle and Jensen inequalities (as in the derivation of (A.1))
\[
0 \geq NV_{1NT, \lambda_1}^f (b^f) \geq \sum_{t=1}^T \left[ b_t^f \hat{\Delta}_t^{-1} b_t^f - 2 b_t^f \hat{R}_{nt}^f \right] - N^{1/2} \lambda_1 m_0 \max_{s \in T_{m_0}^0} \hat{\psi}_s \max_{s \in T_{m_0}^0} \sum_{t=1}^T \omega_{ts}^f b_t^s
\]
\[
\geq \sum_{t=1}^T \left[ b_t^f \hat{\Delta}_t^{-1} b_t^f - \left( 2 \right) \left\| \hat{R}_{nt}^f \right\| + N^{1/2} \lambda_1 (m_0)^{3/2} \max_{s \in T_{m_0}^0} \hat{\psi}_s \max_{s \in T_{m_0}^0} \left\| \omega_{ts}^f \right\| \left\| b_t^f \right\| \right]
= \sum_{t=1}^T \left[ b_t^f \hat{\Delta}_t^{-1} b_t^f - O_P (1) \left\| b_t^f \right\| \right].
\]
It follows that $0 \geq N \left[ v_{1 \text{INT,L}_1}^T ( \tilde{b}^1 ) - v_{1 \text{INT,L}_1}^T ( 0_{T^0 p+1} ) \right] \geq \sum_{t=1}^{T} \left[ \tilde{b}^l_t \Delta_t \tilde{b}^l_t - O_P(1) \right] \left\| \tilde{b}^l_t \right\| \text{ for each } t$. Otherwise, $\{\tilde{b}^l_t\}$ cannot minimize $v_{1 \text{INT,L}_1}^T ( \tilde{b}^1 )$. This implies that $\tilde{b}^l_t = N^{1/2} (\beta^l_t - \theta^0_t) = O_P(1)$ by the same arguments as used in the proof of Lemma A.1.

**Proof of Theorem 3.2.** We want to demonstrate that

$$P \left( \left\| \tilde{\theta}_t \right\| = 0 \text{ for all } t \in T_{m^0}^0 \right) \to 1 \text{ as } N \to \infty. \quad (A.2)$$

Suppose that to the contrary, $\tilde{\theta}_t = \tilde{\beta}_t - \tilde{\beta}_{t-1} \neq 0$ for some $t \in T_{m^0}^0$ for sufficiently large $N$ or $(N,T)$. Then there exists $r \in \{1, \ldots, p\}$ such that $\left\| \tilde{\theta}_{t,r} \right\| = \max \{ \left\| \tilde{\theta}_{t,l} \right\|, l = 1, \ldots, p \}$, where for any $p \times 1$ vector $a_t$, $a_{t,l}$ denotes its $l$th element. Without loss of generality (wlog) assume that $r = p$, implying that $\left\| \tilde{\theta}_{t,p} \right\| / \left\| \tilde{\theta}_t \right\| \geq 1/\sqrt{p}$. To consider the first order condition (FOC) with respect to (wrt) $\beta_t$, $t \geq 2$, based on subdifferential calculus (e.g., Bereskin (1995, Appendix B.5)), we distinguish two cases: (a) $2 \leq t \leq T - 1$ and (b) $t = T$ and $T \in T_{m^0}^0$.

In case (a), we consider two subcases: (a1) $t+1 = T^0_j \in T_{m^0}^0$ for some $j = 1, \ldots, m^0$, and (a2) $t+1 \in T_{m^0}^0$. In either case, we can apply the FOC wrt $\beta_{t,p}$ and the equality $\Delta \tilde{\theta}_t = \beta^0_{t} x_{it} - \beta^0_{t-1} x_{i,t-1} + \Delta u_{it}$ to obtain

\begin{align*}
0 &= -\frac{2}{\sqrt{N}} \sum_{i=1}^{N} \left( \Delta y_{it} - \tilde{\beta}_{t} x_{it} - \tilde{\beta}^0_{t-1} x_{i,t-1} \right) x_{it,p} + \frac{2}{\sqrt{N}} \sum_{i=1}^{N} \left( \Delta y_{it} - \tilde{\beta}^0_{t-1} x_{i,t-1} + \tilde{\beta}^0_{t} x_{it} \right) x_{it,p} \\
&+ \sqrt{N} \lambda_1 \tilde{w}_t \left\| \tilde{\theta}_t \right\| - \sqrt{N} \lambda_1 \tilde{w}_{t+1} e_{t+1,p} \\
&= -\frac{2}{\sqrt{N}} \sum_{i=1}^{N} \left[ \left( \tilde{\beta}_{t+1} - \tilde{\beta}^0_{t+1} \right)' x_{i,t+1} - 2 \left( \tilde{\beta}_{t} - \tilde{\beta}^0_{t} \right)' x_{it} + \left( \tilde{\beta}^0_{t-1} - \tilde{\beta}^0_{t} \right)' x_{i,t-1} \right] x_{it,p} \\
&+ \frac{2}{\sqrt{N}} \sum_{i=1}^{N} \Delta^2 u_{i,t+1} x_{it,p} + \sqrt{N} \lambda_1 \tilde{w}_t \left\| \tilde{\theta}_t \right\| - \sqrt{N} \lambda_1 \tilde{w}_{t+1} e_{t+1,p} \\
&= B_{1t} + B_{2t} + B_{3t} - B_{4t}, \text{ say,}
\end{align*}

where $e_{t+1} = \tilde{e}_{t+1} / \left\| \tilde{e}_{t+1} \right\|$ if $\left\| \tilde{e}_{t+1} \right\| \neq 0$ and $\left\| e_{t+1,p} \right\| \leq 1$ otherwise, $e_{t+1,p}$ is the $p$th element in $e_{t+1}$. By Assumptions A.1(i)-(ii) and Theorem 3.1, $B_{1t} = O_P(1)$ and $B_{2t} = O_P(1)$. In view of the fact that $\tilde{w}_{t+1} = O_P(N^{-\epsilon_1})$ for $t \in T^0_{m^0}$, $|B_{3t}| \geq \sqrt{N} \lambda_1 \tilde{w}_t / \sqrt{p}$, which is explosive in probability under Assumption A.2(iii) (i.e., $N^{(\epsilon_1 + 1)/2} \lambda_1 \to \infty$).

To bound the probability order of $B_{4t}$, we distinguish two subcases. In subcase (a1), noting that $\tilde{\beta}_{t+1} - \tilde{\beta}_t \overset{P}{=} \beta^0_{t+1} \neq 0$ by Theorem 3.1, we have $\tilde{w}_{t+1} = \| \beta^0_{t+1} + O_P(N^{-1/2}) \|^{-\epsilon_1} = O_P(J_{m^0}^{-\epsilon_1})$ and $B_{4t} = \sqrt{N} \lambda_1 \tilde{w}_{t+1} e_{t+1,p} = O_P(\sqrt{N} \lambda_1 J_{m^0}^{-\epsilon_1}) = O_P(1)$. Consequently, $|B_{3t}| \gg |B_{1t} + B_{2t} - B_{4t}|$ so that (A.3) cannot be true for sufficiently large $N$ or $(N,T)$. Then we conclude that w.p.a.1, $\tilde{\theta}_t$ must be in a position where $\left\| \tilde{\theta}_t \right\|$ is not differentiable in subcase (a1). In addition, a direct implication of this result is that if $t = T^0_j - 1 \in T^0_{m^0}$ for some $j = 1, \ldots, m^0$, then $P \left( \left\| \tilde{\theta}_{j-1} \right\| = 0 \right) \to 1$ as $N \to \infty$ and $\sqrt{N} \lambda_1 \tilde{w}_{T^0_j-1} e_{T^0_j-1} = O_P(1)$ in order for the FOC to hold for $t = T^0_j - 1$.
In subcase (a2), difficulty arises as \( \dot{w}_{t+1} = O_{p}(N^{\alpha_3}/2) \) and \( \sqrt{N} \lambda_1 \dot{w}_{t+1} = O_{p}(N^{(1+\alpha_3)/2} \lambda_1) \). But we can apply the implication from the result in subcase (a1) recursively. When \( t = T_j^0 - 2 \in T_{m_0}^0 \) for some \( j = 1, \ldots, m_0 \), \( B_{mt} = \sqrt{N} \lambda_1 \dot{w}_{T_j^0 - 1}e_{T_j^0 - 1,p} = O_{p}(1) \) and \( |B_{mt}| \gg |B_{mt} + B_{2t} - B_{4t}| \). Thus (A.3) cannot hold for \( t = T_j^0 - 2 \in T_{m_0}^0 \) either and we must have \( \| \theta_{T_j^0 - 2} \| \rightarrow 1 \) as \( N \rightarrow \infty \) and \( \sqrt{N} \lambda_1 \dot{w}_{T_j^0 - 2}e_{T_j^0 - 2} = O_{p}(1) \) in order for the FOC to hold for \( t = T_j^0 - 2 \). Deducting in this way until we reach \( t = T_j^0 - 1 \in T_{m_0}^0 \). Consequently, \( \tilde{\theta}_t \) must be in a position that \( \| \tilde{\theta}_t \| \) is not differentiable for all \( t \in T_{m_0}^0 \) and \( t \neq T \).

In case (b), noting only one term in the penalty term \( (\lambda_1 \sum_{i=2}^{T} \dot{w}_i \| \beta_t - \beta_{t-1} \|) \) is involved with \( \beta_T \), it is easy to show that \( \tilde{\theta}_T = \tilde{\beta}_T - \tilde{\beta}_{T-1} \) must be in a position where \( \| \tilde{\theta}_T \| \) is not differentiable if \( T \in T_{m_0}^0 \). Consequently (A.2) follows.

**Proof of Corollary 3.3.** We consider two cases: (a) \( t \in T_{m_0}^0 \), and (b) \( t \in T_{m_0}^0 \). In case (a), Theorem 3.2 implies that asymptotically no time point in \( T_{m_0}^0 \) can be identified as an estimated break date so that \( \tilde{m} \leq m^0 \). In case (b), we want to show that all break points in \( T_{m_0}^0 \) must be identified as an estimated break point. Suppose not. Then there exists \( t \in T_{m_0}^0 \) such that \( \| \tilde{\theta}_t \| = 0 \). By the \( \sqrt{N} \)-consistency of \( \tilde{\theta}_t \) and the fact \( \tilde{\theta}_t = \tilde{\beta}_t - \tilde{\beta}_{t-1} = \beta^0_t - \beta^0_{t-1} + O_{p}(N^{-1/2}) = \theta^p_t + O_{p}(N^{-1/2}) \) by Theorem 3.1, we have \( \| \tilde{\theta}_t \| = O(N^{-1/2}) \), which contradicts the assumption that \( N^{1/2}J_{min} \rightarrow \infty \) as \( N \rightarrow \infty \) as \( \| \tilde{\theta}_t \| \geq J_{min} \) for any \( t \in T_{m_0}^0 \).

**Proof of Theorem 3.4.** Note that \( \tilde{\alpha}^p_m(\tilde{T}_m) = (\tilde{\alpha}^p_{m1}(\tilde{T}_m), \ldots, \tilde{\alpha}^p_{m+1}(\tilde{T}_m))' = \arg \min_{\alpha} Q_{NT}\left(\alpha_m; \tilde{T}_m\right) \). The FOCs for this minimization problem are

\[
0_{p\times 1} = -\frac{2}{N} \sum_{i=1}^{m+1} \sum_{t=1}^{T} (\Delta y_{it} - \hat{\alpha}^p_{i1} \Delta x_{it}) \Delta x_{it} + \frac{2}{N} \sum_{i=1}^{m} (\Delta y_{it} - \hat{\alpha}^p_{i1} x_{it} \tilde{T}_m + \hat{\alpha}^p_{i1} x_{i,T_1} - \hat{\alpha}^p_{i1} x_{i,T_1}) x_{i,T_1} - \frac{2}{N} \sum_{i=1}^{m} (\Delta y_{it} - \hat{\alpha}^p_{i1} x_{i,T_1} + \hat{\alpha}^p_{i1} x_{i,T_1} x_{i,T_1} - \hat{\alpha}^p_{i1} x_{i,T_1} x_{i,T_1}) x_{i,T_1} - \frac{2}{N} \sum_{i=1}^{m} (\Delta y_{it} - \hat{\alpha}^p_{i1} x_{i,T_1} x_{i,T_1} + \hat{\alpha}^p_{i1} x_{i,T_1} x_{i,T_1}) x_{i,T_1}
\]

where we suppress the dependence of \( \hat{\alpha}^p_{i1} \) on \( \tilde{T}_m \). Let \( \tilde{P}_{a,l} = \frac{1}{N} \sum_{i=1}^{N} (\tilde{\alpha}^p_{i1} - \hat{\alpha}^p_{i1} \tilde{\alpha}^p_{i1} + \hat{\alpha}^p_{i1} \tilde{\alpha}^p_{i1})' \tilde{t}_{i1} + \hat{\alpha}^p_{i1} \tilde{\alpha}^p_{i1} \tilde{t}_{i1} \tilde{t}_{i1} \) and \( \tilde{P}_{a,l} = \frac{1}{N} \sum_{i=1}^{N} \tilde{t}_{i1} \tilde{t}_{i1} \tilde{t}_{i1} \tilde{t}_{i1} \). One can readily solve for \( \tilde{\alpha}^p_{i1} \) to obtain \( \tilde{\alpha}^p_{i1} = \tilde{P}_{a,l}^{-1} \tilde{Y}^p_{a,l+1} \), where

\[
\tilde{P}_{a,l}^m = \text{TriD} \left( \tilde{P}_{a,l}^m, \tilde{T}_{m_0} \right), \quad \tilde{Y}_{a,l}^m = \left( \tilde{P}_{a,l}^m \tilde{\alpha}^p_{m1}, \tilde{\alpha}^p_{m2} \right)^{1/2}, \quad \tilde{P}_{a,l}^m = \text{TriD} \left( \tilde{P}_{a,l}^m \tilde{\alpha}^p_{m1}, \tilde{\alpha}^p_{m2} \right)^{1/2}, \quad \tilde{P}_{a,l}^m = \text{TriD} \left( \tilde{P}_{a,l}^m \tilde{\alpha}^p_{m1}, \tilde{\alpha}^p_{m2} \right)^{1/2}, \quad \tilde{P}_{a,l}^m = \text{TriD} \left( \tilde{P}_{a,l}^m \tilde{\alpha}^p_{m1}, \tilde{\alpha}^p_{m2} \right)^{1/2}, \quad \tilde{P}_{a,l}^m = \text{TriD} \left( \tilde{P}_{a,l}^m \tilde{\alpha}^p_{m1}, \tilde{\alpha}^p_{m2} \right)^{1/2}.
\]
\[ \tilde{\Phi}_1 = \tilde{\Phi}_1 + \phi_{xx, \tilde{T}_1}, \quad \tilde{\Phi}_l = \tilde{\Phi}_l + \phi_{xx, \tilde{T}_l-1} \quad \text{for } l = 2, \ldots, \tilde{m}, \quad \tilde{\Phi}_{\tilde{m}+1} = \tilde{\Phi}_{\tilde{m}+1} + \phi_{xx, \tilde{T}_{\tilde{m}}}, \]

and \[ \tilde{\Phi}_{\tilde{m}+1} = \phi_{xx, \tilde{T}_{\tilde{m}+1}} \quad \text{for } l = 1, \ldots, \tilde{m}. \]

By Corollary 3.3, \[ \tilde{\alpha}_{\tilde{m}}(\tilde{T}_{\tilde{m}}) = \tilde{\alpha}_{\tilde{m}}(T_{\tilde{m}}) \] w.p.a.1. Therefore we can study the asymptotic distribution of \( \tilde{\alpha}_{\tilde{m}}(\tilde{T}_{\tilde{m}}) \) by studying that of \( \tilde{\alpha}_{\tilde{m}}(T_{\tilde{m}}) \). Note that \[ \tilde{\alpha}_{\tilde{m}}(T_{\tilde{m}}) = \Phi^{-1}_{\tilde{m}} \Psi_{\tilde{m}}^\psi, \]

where \( \Psi_{\tilde{m}} \) is defined in (3.1) and (3.2), respectively. It is easy to verify that

\[ \sqrt{N}D_{\tilde{m}+1} (\tilde{\alpha}_{\tilde{m}}(T_{\tilde{m}}) - \alpha^0) = \left( D_{\tilde{m}+1}^{-1} \Phi_{\tilde{m}} D_{\tilde{m}+1}^{-1} \right)^{-1} \sqrt{N}D_{\tilde{m}+1}^{-1} \Psi_{\tilde{m}}^\psi \]

where \( \Psi_{\tilde{m}} \) is defined in (3.2). Then by Assumption A.3(i), \( D_{\tilde{m}+1}^{-1} \Phi_{\tilde{m}} D_{\tilde{m}+1}^{-1} \overset{a.s.}{\rightarrow} \Phi_0 > 0 \). By Assumption A.3(ii), \( \sqrt{N}D_{\tilde{m}+1}^{-1} \Psi_{\tilde{m}}^\psi \overset{D}{\rightarrow} N(0, \Omega_0) \). Then by the Slutsky lemma, \( \sqrt{N}D_{\tilde{m}+1} (\tilde{\alpha}_{\tilde{m}}(T_{\tilde{m}}) - \alpha^0) \overset{D}{\rightarrow} N(0, \Phi_0^{-1} \Omega_0 \Phi_0^{-1}) \). This completes the proof of the theorem.

**Proof of Theorem 3.5.** Recall \( \tilde{\alpha}_{\tilde{m}}(T_{\tilde{m}}) = (\tilde{\alpha}(T_{\tilde{m}})_1, \ldots, \tilde{\alpha}(T_{\tilde{m}})_m) \) denotes the set of post-Lasso OLS estimates of the regression coefficients based on the break dates in \( T_{\tilde{m}} = (T_1(\lambda), \ldots, T_m(\lambda)) \), where we make the dependence of various estimates on \( \lambda \) explicit. Let \( \tilde{\sigma}_{\tilde{m}}^2 = \frac{1}{T-1} \sum_{j=1}^{T} (\tilde{\alpha}(T_{\tilde{m}})_j - \tilde{\alpha}(T_{\tilde{m}})_0)^2 \). Then by the Slutsky lemma, \( \sqrt{N}D_{\tilde{m}+1} (\tilde{\alpha}_{\tilde{m}}(T_{\tilde{m}}) - \alpha^0) \overset{D}{\rightarrow} N(0, \Phi_0^{-1} \Omega_0 \Phi_0^{-1}) \). This completes the proof of the theorem.

**Case 1:** Under-fitted model: \( \tilde{m}_{\lambda_1} < m \). By Lemma 4.2 below, inf_{\lambda_1} \tilde{\sigma}_{\tilde{m}_{\lambda_1}}^2 - \tilde{\sigma}_{\tilde{m}^0}^2 \geq c_0 \) where \( c_0 = \frac{\inf_{T-1} \tilde{m}_{\lambda_1} [c + o_p(1)]}{T-1} \) for some \( c > 0 \). Then by Assumption A.5,

\[ P \left( \inf_{\lambda_1} IC(\lambda_1) > IC(\lambda^0_{NT}) \right) = P \left( \inf_{\lambda_1} \left[ \left( \tilde{\sigma}_{\tilde{m}_{\lambda_1}}^2 - \tilde{\sigma}_{\tilde{m}^0}^2 \right) + \rho_{1NT} (\tilde{m}_{\lambda_1} - m) \right] > 0 \right) \]

\[ \geq P \left( \frac{\inf_{T=1} \tilde{m}_{\lambda_1} [c + o_p(1)]}{\rho_{1NT} (T-1)} + o_p(1) > 1 \right) \]

**Case 2:** Over-fitted model: \( \tilde{m}_{\lambda_1} > m \). Let \( T_m = \{ T_m = \{ T_1, \ldots, T_m \} : 2 \leq T_1 < \ldots < T_m \leq T \} \). Given \( T_m = \{ T_1, \ldots, T_m \} \), \( T_{m^*} = \{ T_1, T_2, \ldots, T_{m^*} \} \) denote the union of \( T_m \) and \( T_{m^*} \) with
elements ordered in non-descending order: \(2 \leq \tilde{T}_1 < \tilde{T}_2 < \cdots < \tilde{T}_{m^* + m^0} \leq T\) for some \(m^* \in \{0, 1, \ldots, m\}\).

Let \(\tilde{\alpha}_m^p(T_m) \equiv (\tilde{\alpha}_T^m(T_m), \ldots, \tilde{\alpha}_{m+1}^m(T_m))' = \arg \min_{\alpha_m} Q_{1NT}(\alpha_m; T_m)\) and \(\tilde{\sigma}_m^2 \equiv Q_{1NT}(\tilde{\alpha}_m^p(T_m); T_m)\). \(\tilde{\sigma}_m^2\) is analogously defined. In view of the fact that \(\tilde{\sigma}_{T_{m^* + m^0}}^2 \leq \tilde{\sigma}_{T_{m^0}}^2\) for all \(T_m \in T_m\), \(N(\tilde{\sigma}_{T_{m^* + m^0}}^2 - \tilde{\sigma}_{T_{m^0}}^2) = O_P(1)\) uniformly in \(T_m \in T_m\) by Lemma A.3 below, and \(N \rho_{1NT} \to \infty\) by Assumption A.5, we have

\[
P \left( \inf_{\lambda_1 \in 1T} IC(\lambda_1) > IC(\lambda_{1NT}^0) \right) \\
\geq P \left( \min_{m^0 < m \leq m_{\max}} \inf_{T_m \in T_m} \left[ N \left( \tilde{\sigma}_{T_{m^0}}^2 - \tilde{\sigma}_{T_{m^0}}^2 \right) + N \rho_{1NT} (m - m^0) \right] \right) > 0 \\
\geq P \left( \min_{m^0 < m \leq m_{\max}} \inf_{T_m \in T_m} \left[ N \left( \tilde{\sigma}_{T_{m^* + m^0}}^2 - \tilde{\sigma}_{T_{m^0}}^2 \right) + N \rho_{1NT} (m - m^0) \right] \right) > 0 \\
\to 1 \text{ as } N \to \infty.
\]

**Lemma A.2** Let \(T_m = \{T_m = \{T_1, \ldots, T_m\} : 2 \leq T_1 < \cdots < T_m \leq T, T_0 = 1 \text{ and } T_{m+1} = T + 1\}.\) Then \(\min_{m^0 \leq m \leq m_{\max}} \inf_{T_m \in T_m} \frac{(T-1)}{\min_{\alpha_m} \min T_m} (\tilde{\sigma}_{T_{m^0}}^2 - \tilde{\sigma}_{T_{m^0}}^2) \geq c + o_P(1)\) for some \(c > 0\).

**Proof.** First, by the results for least squares regressions, we can readily show that \(\tilde{\sigma}_{T_{m^0}}^2 = \tilde{\sigma}_{N_1T}^2 + O_P((NI_{\min})^{-1}).\) In view of the fact that that \(\tilde{\alpha}_m^p = \arg \min_{\alpha_m} D_{1NT}(\alpha_m; T_m)\) where

\[
D_{1NT}(\alpha_m; T_m) = Q_{1NT}(\alpha_m; T_m) - Q_{1NT}(\alpha_{m^0}^0; T_{m^0}^0)
\]

\[
= \frac{1}{N} \sum_{m+1} \sum_{j=1}^{T_j-1} \sum_{i=1}^{N} \left[ (\Delta y_{it} - \alpha_j x_{it})^2 - (\Delta u_{it})^2 \right]
\]

\[
+ \frac{1}{N} \sum_{j=1}^{m} \sum_{i=1}^{N} \left[ (\Delta y_{i(T_j)} - \alpha_j + x_{i(T_j)} + \alpha_j x_{i(T_j-1)})^2 - (\Delta u_{i(T_j)})^2 \right],
\]

and \((T-1)(\tilde{\sigma}_{T_{m^0}}^2 - \tilde{\sigma}_{N_1T}^2) = D_{1NT}(\tilde{\alpha}_m^p(T_m); T_m)\), it suffices to show that \(\frac{1}{\min_{\alpha_m} \min T_m} D_{1NT}(\tilde{\alpha}_m^p(T_m); T_m) \geq c + o_P(1)\) uniformly in \(T_m \in T_m\) for each \(m \in \{0, m^0 - 1\}\). We consider three cases: (a) \(m^0 = 1\), (b) \(m^0 = 2\), and (c) \(3 < m^0 \leq m_{\max}\).

In case (a), \(m = 0\) and \(T_m = T_0\) becomes empty so that the post Lasso estimate \(\tilde{\alpha}_m^p(T_m) = \tilde{\alpha}_m^p(T_0) = \tilde{\alpha}_1^p(T_0)\) becomes the OLS estimate in the first-differenced model: \(\hat{\alpha}_1^p(T_0) = \left( \sum_{t=2}^T \sum_{i=1}^N \Delta x_{it} (\Delta x_{it})' \right)^{-1} \times \sum_{t=2}^T \sum_{i=1}^N \Delta x_{it} \Delta y_{it}\). Noting that

\[
\Delta y_{it} = x_{it}' \beta_1^0 - x_{it-1}' \beta_1^0 + \Delta u_{it} =
\begin{cases}
(\Delta x_{it})' \alpha_1^0 + \Delta u_{it} & \text{if } 2 \leq t \leq T_1 - 1 \\
(\Delta x_{it})' \alpha_1^0 + \Delta u_{it} & \text{if } t = T_1 \text{ or } t = T_1 + 1 \\
(\Delta x_{it})' \alpha_2^0 + \Delta u_{it} & \text{if } T_1 + 1 \leq t \leq T
\end{cases}
\]

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we have

\[
\hat{\alpha}_1^p(T_0) = M_{NT}^{-1} M_{NT} \alpha_1^0 + M_{NT}^{-1} M_{2NT} \alpha_2^0 + M_{NT}^{-1} \frac{1}{N(T-1)} \sum_{i=1}^{N} \Delta x_i \theta_i^p \left( x_i \theta_i^p \alpha_2^0 - x_i \theta_i^p \alpha_1^0 \right) \\
+ M_{NT}^{-1} \frac{1}{N(T-1)} \sum_{t=2}^{T} \sum_{i=1}^{N} \Delta x_{it} \Delta u_{it} \\
= M_{NT}^{-1} M_{NT} \alpha_1^0 + M_{NT}^{-1} M_{2NT} \alpha_2^0 + M_{NT}^{-1} \frac{1}{T-1} \left( \phi_{\Delta x_x.T_0} \alpha_2^0 - \phi_{\Delta x_x.T_0} \alpha_1^0 \right) \\
+ O_P \left( (N(T-1)^{-1/2} \right) 
\]

where \( M_{NT} = \frac{1}{N(T-1)} \sum_{t=2}^{T} \sum_{i=1}^{N} \Delta x_{it} (\Delta x_{it})', \ M_{1NT} = \frac{1}{N(T-1)} \sum_{t=2}^{T} \sum_{i=1}^{N} \Delta x_{it} (\Delta x_{it})', \ M_{2NT} = \frac{1}{N(T-1)} \sum_{t=2}^{T} \sum_{i=1}^{N} \Delta x_{it} (\Delta x_{it})', \) and the last line follows from Assumption A.1.\(^9\) Note that

\[
D_{1NT}(\hat{\alpha}_0^p(T_0); T_0) = \frac{1}{N} \sum_{t=2}^{T} \sum_{i=1}^{N} \left[ (\Delta y_{it} - \hat{\alpha}_1^p(T_0))' \Delta x_{it} \right]^2 \\
= \frac{1}{N} \sum_{t=2}^{T} \sum_{i=1}^{N} \left[ (\beta_0^p(T_0))' \Delta x_{it} \right]^2 \\
+ 2 \frac{1}{N} \sum_{t=2}^{T} \sum_{i=1}^{N} \left[ (\beta_0^p(T_0))' \Delta x_{it} \right]^2 \\
\equiv D_1 + 2D_2, \text{ say.}
\]

Further,

\[
D_1 = \frac{1}{N} \sum_{t=2}^{T} \sum_{i=1}^{N} \left[ (\alpha_1^0 - \hat{\alpha}_1^p(T_0))' \Delta x_{it} \right]^2 \\
+ \frac{1}{N} \sum_{t=2}^{T} \sum_{i=1}^{N} \left[ (\alpha_1^0 - \hat{\alpha}_1^p(T_0))' \Delta x_{it} \right]^2 \\
+ \frac{1}{N} \sum_{t=2}^{T} \sum_{i=1}^{N} \left[ (\alpha_1^0 - \hat{\alpha}_1^p(T_0))' \Delta x_{it} \right]^2 \\
= (T-1) \left[ (\alpha_1^0 - \hat{\alpha}_1^p(T_0))' M_{1NT} [\alpha_1^0 - \hat{\alpha}_1^p(T_0)]' \right] \\
+ (T-1) \left[ (\alpha_1^0 - \hat{\alpha}_1^p(T_0))' M_{2NT} [\alpha_1^0 - \hat{\alpha}_1^p(T_0)]' \right] \\
+ \frac{1}{N} \sum_{t=2}^{T} \sum_{i=1}^{N} \left[ (\alpha_1^0 - \hat{\alpha}_1^p(T_0))' \Delta x_{it} \right]^2 \\
\equiv D_{11} + D_{12} + D_{13}, \text{ say.}
\]

Let \( d_1^0 = \alpha_2^0 - \alpha_1^0. \) Then \( \hat{\alpha}_1^p(T_0) - \alpha_1^0 = M_{NT}^{-1} M_{2NT} d_1^0 + O_P \left( (N(T-1)^{-1/2} \right) \) and \( \hat{\alpha}_1^p(T_0) - \alpha_2^0 = -M_{NT}^{-1} M_{1NT} d_1^0 + O_P \left( (N(T-1)^{-1/2} \right), \) where \( M_{2NT} = M_{2NT} + \frac{1}{N(T-1)^{-1/2}} \phi_{\Delta x_x.T_0} \) and \( M_{1NT} = M_{1NT} - \frac{1}{N(T-1)^{-1/2}} \phi_{\Delta x_x.T_0}, \phi_{\Delta x_x.T_0}. \)

Noting that \( \| d_1^0 \| = J_{\min}, \phi_{\Delta x_x.T_0} = O_P (1), \phi_{\Delta x_x.T_0} = O_P (1), M_{NT}^1 = O_P (1), M_{NT} = O_P \left( \frac{1}{T-1} \right), M_{2NT} = O_P \left( \frac{1}{T-1} \right), M_{1NT} = O_P \left( \frac{1}{T-1} \right), \) we have

\[
d_1^0 M_{2NT}^{-1} M_{NT}^{-1} M_{NT}^{-1} M_{2NT} d_1^0 \geq \frac{J^2_{\min} (T^2 - 1) (I_j^0 - 1)^2}{(T-1)^3} \theta_1^p,
\]

\(^9\)Strictly speaking, we need \( 3 \leq T_0^0 \leq T - 1. \) If \( T_0^0 = 2 (\text{resp. } T), \) then \( M_{1NT} = 0 (\text{resp. } M_{2NT} = 0) \) as \( \sum_{t=a}^{b} \equiv 0 \) when \( a > b. \)
and

\[ d_1'' M_{1NT}^{-1} M_{2NT}^{-1} M_{1NT}^{-1} \frac{M_{1NT} d_1'}{J_{min}} \geq \frac{J_{2min} (I_0^1 - 1)^2 (I_0^2 - 1)}{(T - 1)^3} \hat{c}_{2NT}, \]

where \( \hat{c}_{1NT} = \frac{(T-1)^3}{(I_0^1 - 2)(I_0^2 - 2)} \lambda_{min} (M_{2NT}^{-1} M_{1NT}^{-1} M_{2NT}^{-1} M_{1NT}) \), and \( \hat{c}_{2NT} = \frac{(T-1)^3}{(I_0^1 - 2)(I_0^2 - 2)} \lambda_{min} (M_{1NT}^{-1} M_{2NT}^{-1} M_{1NT}^{-1} M_{2NT}) \). To bound \( D_1 \), we consider two subcases: (a1) \( I_0^1 \geq 2 \) and \( I_0^2 \geq 2 \), and (a2) \( I_0^1 \) or \( I_0^2 \) = 1. Observe that in subcase (a1)

\[
\frac{D_{11} + D_{12}}{I_{min} J_{min}^2} = \frac{T - 1}{I_{min} J_{min}^2} \left\{ \left[ \alpha_1^0 - \tilde{\alpha}_1^0 (T_0) \right]' M_{1NT} \left[ \alpha_1^0 - \tilde{\alpha}_1^0 (T_0) \right]' M_{2NT} \left[ \alpha_2^0 - \tilde{\alpha}_2^0 (T_0) \right]' M_{2NT} \left[ \alpha_2^0 - \tilde{\alpha}_2^0 (T_0) \right]' \right\}
\]

\[
\geq \frac{T - 1}{I_{min} J_{min}^2} \left\{ \frac{J_{2min}^2 (I_0^1 - 1) (I_0^2 - 1)^2}{(T - 1)^3} \hat{c}_{1NT} + \frac{J_{2min}^2 (I_0^1 - 1) (I_0^2 - 1)^2}{(T - 1)^3} \hat{c}_{2NT} \right\}
+
\frac{T - 1}{I_{min} J_{min}^2} \lambda_{min} (M_{1NT}^{-1} M_{2NT}^{-1} M_{1NT}^{-1} M_{2NT})
\]

\[
\geq \frac{1}{I_{min}} \left\{ \frac{(I_0^1 - 1) (I_0^2 - 1)^2}{(T - 1)^2} + \frac{(I_0^1 - 1) (I_0^2 - 1)^2}{(T - 1)^2} \right\} \hat{c}_{NT} + o_P(1)
\]

\[
= \frac{(I_0^1 - 1) (I_0^2 - 1) (T - 2)}{I_{min} (T - 1)^2} \hat{c}_{NT} + o_P(1)
\]

where \( \hat{c}_{NT} = min(\hat{c}_{1NT}, \hat{c}_{2NT}) \) and s.m. denotes terms that are of smaller order than the expressed terms. Then \( \frac{1}{I_{min} J_{min}^2} D_1 \geq \hat{c} + o_P(1) \) where \( \hat{c} = \lim_{T \to \infty} \frac{(I_0^1 - 1) (I_0^2 - 1) (T - 2)}{I_{min} (T - 1)^2} \lambda_{min} (M_{1NT}^{-1} M_{2NT}^{-1} M_{1NT}^{-1} M_{2NT}) \) > 0. In subcase (a2), \( D_{11} = 0 \) or \( D_{12} = 0 \), \( (D_{11} + D_{12}) / (I_{min} J_{min}^2) = o_P(1) \), and we need to show that \( \frac{1}{I_{min} J_{min}^2} D_{13} \) is stochastically bounded below by a positive constant. By Assumption A.4(i),

\[
\frac{D_{13}}{I_{min} J_{min}^2} \geq \frac{1}{I_{min} J_{min}^2} \min_{\alpha_1} \frac{1}{N} \sum_{i=1}^{N} \left( \alpha_2^0 - \alpha_1 \right)' x_i T_0^1 - \left( \alpha_1 - \alpha_1 \right)' x_i T_0^2 - 1 \right]^2 \geq \frac{\hat{c}_\alpha}{I_{min}} + o_P(1).
\]

Then \( \frac{1}{I_{min} J_{min}^2} D_1 \geq \frac{\hat{c}_\alpha}{I_{min}} + o_P(1) \).

To determine the probability order of \( D_2 \), we make the following decomposition:

\[
D_2 = [\alpha_1^0 - \tilde{\alpha}_1^0 (T_0)]' M_{1NT}^u [\alpha_2^0 - \tilde{\alpha}_2^0 (T_0)]' M_{2NT}^u
\]

\[ + \frac{1}{N} \sum_{i=1}^{N} \left\{ \left[ \alpha_2^0 - \tilde{\alpha}_2^0 (T_0) \right]' x_i T_0^1 - \left[ \alpha_1^0 - \tilde{\alpha}_1^0 (T_0) \right]' x_i T_0^2 - 1 \right\} \Delta u_{iT_0}^0
\]

\[ = D_{21} + D_{22} + D_{23}, \text{ say,}
\]

where \( M_{1NT}^u = \frac{1}{N} \sum_{t=T_0}^{T_0-1} \sum_{i=1}^{N} \Delta u_{it} \Delta u_{it} \), and \( M_{2NT}^u = \frac{1}{N} \sum_{t=T_0+1}^{T} \sum_{i=1}^{N} \Delta u_{it} \Delta u_{it} \). Noting that \( \alpha_0^0 - \tilde{\alpha}_0^0 (T_0) = J_{min} OP \left( \frac{I_0^0 - 1}{T_0 - 1} \right), \alpha_2^0 - \tilde{\alpha}_2^0 (T_0) = J_{min} OP \left( \frac{I_0^2 - 1}{T_0 - 1} \right), M_{1NT}^u = OP \left( \sqrt{I_0^1 - 1} / N \right), \) and \( M_{2NT}^u = OP \left( \sqrt{I_0^2 - 1} / N \right), \) we have

\[
D_{21} + D_{22} = \frac{1}{I_{min} J_{min}^2} \left[ \left( I_0^0 - 1 \right) \sqrt{I_0^1 - 1} + \left( I_0^1 - 1 \right) \sqrt{I_0^2 - 1} \right] = o_P(1).
\]
Similarly, noting \( \frac{1}{N} \sum_{i=1}^{N} x_{it} \Delta u_{it(T)} = O_P \left( N^{-1/2} \right) \) for \( t = T_1^0 \) and \( T_1^0 - 1 \), we have
\[
\frac{D_{23}}{I_{\min} J_{2_{\min}}^2} = \frac{1}{I_{\min} J_{\min} \sqrt{N}} \left\{ O_P \left( \frac{T_1^0 - 1}{T - 1} \right) + O_P \left( \frac{T_2^0 - 1}{T - 1} \right) \right\} = o_P \left( 1 \right).
\]
It follows that \( \frac{D_{23}}{I_{\min} J_{2_{\min}}^2} = o_P \left( 1 \right) \). Consequently, we have \( \frac{1}{I_{\min} J_{2_{\min}}^2} D_{1NT} \left( \hat{\alpha}_m^0 (T_m); T_m \right) \geq \tilde{c} + o_P \left( 1 \right) \).

In cases (b)-(c), it suffices to consider the case where \( m = m^0 - 1 \). If \( m < m^0 - 1 \), one can always augment the set \( T_m \) by \( m^0 - 1 - m \) true break points which are not inside \( T_m \) to make \( D_{1NT} \left( \hat{\alpha}_m^0 (T_m); T_m \right) \) smaller.] For case (b) with \( m = 1 \), we consider three subcases: (b.1) \( 2 \leq T_1 < T_1^0 \), (b.2) \( T_1^0 \leq T_1 \leq T_2^0 \), and (b.3) \( T_2^0 < T_1 \leq T \), where (b.3) is redundant if \( T_2^0 = T \) (i.e., the second true break occurs at the end of the sample). In subcase (b.1), we can focus on the interval \( [T_1 + 1, T] \) which contains two true break points \( T_1^0 \) and \( T_2^0 \) that are not accounted for by the post Lasso estimate \( \hat{\alpha}_1^0 (T_1) = (\hat{\alpha}_1^0 (T_1), \hat{\alpha}_2^0 (T_1))' \). Observe that
\[
D_{1NT} \left( \hat{\alpha}_1^0 (T_1); T_1 \right) = \frac{1}{N} \sum_{i=1}^{N} \left[ (\Delta y_{it} - \hat{\alpha}_1^0 (T_1)') (\Delta x_{it})^2 - (\Delta u_{it})^2 \right]
+ \frac{1}{N} \sum_{i=1}^{T} \sum_{i=1}^{N} \left[ (\Delta y_{it} - \hat{\alpha}_2^0 (T_1)') (\Delta x_{it})^2 - (\Delta u_{it})^2 \right]
+ \frac{1}{N} \sum_{i=1}^{N} \left( (\Delta y_{iT_1} - \hat{\alpha}_2^0 (T_1)') x_{i+T_1} + \hat{\alpha}_1^0 (T_1)' x_{i,T_1-1} \right)^2 - (\Delta u_{iT_1})^2
\equiv \tilde{D}_3 + \tilde{D}_4 + \tilde{D}_5, \text{ say.}
\]
It is easy to show that \( \hat{\alpha}_1^0 (T_1) - \alpha_1^0 = O_P \left( N^{-1/2} \right) \). With this, one can readily show that \( \tilde{D}_3 = O_P \left( N^{-1} \right) \).

Let \( \hat{\alpha} = \text{argmin}_a \frac{1}{N} \sum_{i=1}^{N} (\Delta y_{iT_1} - \alpha x_{iT_1} + \hat{\alpha}_1^0 (T_1)' x_{i,T_1-1})^2 \). By standard results for OLS regressions, \( \hat{\alpha} - \beta_{T_1}^0 = O_P \left( N^{-1/2} \right) \) and
\[
\tilde{D}_5 = \frac{1}{N} \sum_{i=1}^{N} \left( (\Delta y_{iT_1} - \alpha x_{iT_1} + \hat{\alpha}_1^0 (T_1)' x_{i,T_1-1})^2 - (\Delta u_{iT_1})^2 \right) = O_P \left( N^{-1} \right).
\]
It follows that \( \tilde{D}_5 \geq \tilde{D}_5 = O_P \left( N^{-1} \right) \) and \( D_{1NT} \left( \hat{\alpha}_1^0 (T_1); T_1 \right) \geq \tilde{D}_4 + O_P \left( N^{-1} \right) \). A simple repetition of the argument used in case (a) (now with two true breaks) yields \( \frac{1}{I_{\min} J_{2_{\min}}^2} D_{1NT} \left( \hat{\alpha}_1^0 (T_1); T_1 \right) \geq c + o_P \left( 1 \right) \) for some \( c > 0 \). Then by the fact that \( NJ_{2_{\min}}^2 \rightarrow c_J = \infty \), we have \( D_{1NT} \left( \hat{\alpha}_1^0 (T_1); T_1 \right) \geq c + o_P \left( 1 \right) \).

For subcase (b.2), wlog we assume that \( T_1 - T_1^0 \geq T_2^0 - T_1 \), which implies that \( T_1 - T_1^0 \geq I_{\min}/2 \). Then we can focus on the interval \( [2, T_1] \) which contains a true break point \( T_1^0 \). As in subcase (b.1), we can show that \( D_{1NT} \left( \hat{\alpha}_1^0 (T_1); T_1 \right) \geq \tilde{D}_{1NT} + O_P \left( N^{-1} \right) \), where \( \tilde{D}_{1NT} = \frac{1}{N} \sum_{i=1}^{T_1} \sum_{i=1}^{N} \left[ (\Delta y_{iT_1} - \hat{\alpha}_1^0 (T_1)' (\Delta x_{it})^2 - (\Delta u_{it})^2 \right] \).

A simple repetition of the argument used in case (a) yields \( \frac{1}{I_{\min} J_{2_{\min}}^2} D_{1NT} \geq c + o_P \left( 1 \right) \) for some \( c > 0 \).

Subcase (b.3) is analogous to subcase (b.1). Hence, the conclusion follows in subcase (b). Case (c) can be studied analogously. This completes the proof of the lemma. \( \blacksquare \)

**Lemma A.3** Let \( \bar{T}_m = \{T_m = \{T_1, \ldots, T_m\} : T_m \subset T_m, 2 \leq T_1 < \ldots < T_m \leq T \} \) where \( m^0 < m \leq m_{\max} \). Then \( \max_{m^0 < m \leq m_{\max}} \sup_{T_m \in \bar{T}_m} N^{-1} \left| \hat{\gamma}_m^2 - \hat{\gamma}_m^2_{m,0} \right| = O_P \left( 1 \right) \).
Proof. Let \( T_m \in \mathcal{T}_m \) where \( m^0 < m \leq m_{\text{max}} \). In view of the fact that \( \hat{\sigma}_{T_m}^2 \geq \hat{\sigma}_m^2 \) and \( \hat{\sigma}_{T_m}^2 = \hat{\sigma}_m^2 + O_P((NI_{\text{min}})^{-1}) \), we have

\[
0 \leq \hat{\sigma}_{T_m}^2 - \hat{\sigma}_m^2 = \hat{\sigma}_{mT}^2 - \hat{\sigma}_m^2 + O_P((NI_{\text{min}})^{-1}) \leq (m + 1) J_{NT} + O_P((NI_{\text{min}})^{-1}) \quad \text{(A.6)}
\]

where

\[
J_{NT} \equiv \max_{0 < s \leq m} \left( \inf_{(\alpha, \beta)} S_s(\alpha, \beta) \right)
\]

and

\[
S_s(\alpha, \beta) = \frac{1}{N(T-1)} \sum_{t=T+1}^{T+s-1} \sum_{i=1}^{N} \left[ (\Delta y_{it} - \alpha' x_{it})^2 + \beta' x_{it} + \alpha' x_{i,T+s-1} - \beta' x_{i,T+s-1} \right] \]

Let \( \bar{\alpha}_s, \bar{\beta}_s = \arg \min_{(\alpha, \beta)} S_s(\alpha, \beta) \) and \( \bar{\gamma}_s = (\bar{\alpha}_s, \bar{\beta}_s)' \) when \( s < m \) and \( \bar{\alpha}_m = \arg \min_{\alpha} S_m(\alpha) \).

To study \( \min_{(\alpha, \beta)} S_s(\alpha, \beta) \) for \( s = 0, 1, \ldots, m \), we consider three cases: (a) \( T_{s+1} - T_s = 1, s < m \), (b) \( T_{s+1} - T_s = 2, s < m \), and (c) \( T_{s+1} - T_s = 2, s = m \).

In case (a), noting the first term in the definition of \( S_s(\alpha, \beta) \) is zero, we have \( \bar{\gamma}_s = (X_s' X_s)^{-1} X_s' \Delta Y_s \) where \( X_s = (X_{1s}, \ldots, X_{Ns})' \), \( X_{is} = (x_{i,T_s+1-1}, x_{i,T_s+1}', \ldots, x_{i,T_s+1}) \), and \( \Delta Y_s = (\Delta y_{1T_s+1}, \ldots, \Delta y_{NT_s+1})' \). Let \( \bar{\gamma}_s = (\bar{\alpha}_s'^{-1}, \bar{\beta}_s'^{-1})'. \) One can readily show that \( \bar{\gamma}_s - \gamma_s^0 = O_P(N^{-1/2}) \). In addition,

\[
(T - 1) Q_s(\bar{\alpha}_s, \bar{\beta}_s) = \frac{1}{N} \sum_{i=1}^{N} \left( \Delta y_{iT_{s+1}} - \bar{\beta}_s x_{iT_{s+1}} + \bar{\alpha}_s' x_{iT_{s+1}-1} \right)^2 - (\Delta u_{iT_{s+1}})^2.
\]

In case (c), \( \bar{\alpha}_m = \left( \sum_{t=T_{m+1}}^{T_{m}+1} \sum_{i=1}^{N} \Delta x_{it} \Delta y_{it} \right)^{-1} \sum_{t=T_{m+1}}^{T_{m}} \sum_{i=1}^{N} \Delta x_{it} \Delta y_{it} \). It is easy to verify that \( \bar{\alpha}_m - \alpha_{m+1}^0 = O_P(N^{-1/2}) \) and \( \bar{S}_m(\bar{\alpha}_m) = O_P(N^{-1}) \). Similarly, in case (b), one can verify that \( \bar{\gamma}_s - \gamma_s^0 = O_P(N^{-1/2}) \) and \( S_s(\bar{\alpha}_s, \bar{\beta}_s) = O_P(N^{-1}) \). It follows that \( J_{NT} = O_P(N^{-1}) \). This, in conjunction with (A.6), implies that

\[
\hat{\sigma}_{T_m}^2 - \hat{\sigma}_m^2 = O_P(N^{-1})
\]

which holds for all \( m \in \{m^0 + 1, \ldots, m_{\text{max}}\} \) and \( T_m = \{T_1, \ldots, T_m\} \). Then the conclusion follows. \( \blacksquare \)

B Proof of the results in Section 4

Let \( V_{2NT}(\beta) \equiv \sum_{t=2}^{T} \left[ \rho_{it}(\beta_t, \beta_{t-1})' W_t \right] \left[ \frac{1}{N} \sum_{i=1}^{N} \rho_{it}(\beta_t, \beta_{t-1}) \right] \), where \( \rho_{it}(\beta_t, \beta_{t-1}) = z_{it}(\Delta y_{it} - \beta' x_{it} - x_{i,T_s+1} - x_{i,T_s+1}' \beta_{t-1} - 1) \). We first prove a technical lemma.

\footnote{To see why we do not need to consider the case where \( T_{m+1} - T_m = 1 \). Note that if \( s = m \), we have \( T_{m+1} - T_m = T + 1 - T_m \) as \( T_{m+1} = T + 1 \) by default. If \( T_{m+1} - T_m = 1 \), we have \( T_m = T \) so that \( S_m(\alpha) = 0 \) in this case.}
Lemma B.1 Suppose Assumption B.1 holds. Then \( \beta_t^* - \beta_t^0 = O_P \left( N^{-1/2} \right) \) for each \( t = 1, 2, \ldots, T \).

**Proof.** The proof is analogous to that of Lemma A.1 and we only sketch it. Let \( \bar{b}_t = N^{1/2} (\hat{\beta}_t - \beta_t^0) \) and \( \bar{b} = N^{1/2} (\hat{\beta} - \beta^0) \). Let \( \xi_{it} = x'_{it} b_t - x'_{i,t-1} b_{t-1} \) where recall \( b_t = \beta_t - \beta_t^0 \). Noting that \( \Delta y_{it} - x'_{it} \beta_t + x'_{i,t-1} \beta_{t-1} = \Delta u_{it} - N^{-1/2} \xi_{it} \), we have

\[
N \left[ V_{2NT, \lambda_2} (\hat{\beta}) - V_{2NT, \lambda_2} (\beta^0) \right] = \sum_{t=1}^{T} \left\{ \frac{1}{N} \sum_{i=1}^{N} \xi_{it} x'_{it} \right\} \left\{ \frac{1}{N} \sum_{i=1}^{N} z_{it} \xi_{it} \right\} - 2N^{1/2} \sum_{t=1}^{T} \left\{ \frac{1}{N} \sum_{i=1}^{N} \xi_{it} x'_{it} \right\} \left\{ \frac{1}{N} \sum_{i=1}^{N} z_{it} \Delta u_{it} \right\}
\]

\[= b' \tilde{Q}_{NT} b - 2b' \sqrt{N} \tilde{R}_{NT} b \equiv B_1 (b) - 2B_2 (b), \text{ say}, \]

where \( \tilde{Q}_{NT} \) and \( \tilde{R}_{NT} b \) are defined in (2.9) and (2.10), respectively. As in the proof of Lemma A.1, under Assumption B.1, we can readily show that w.p.a.1

\[ T^{-1} \left[ B_1 (b) - 2B_2 (b) \right] \geq \left( \frac{\epsilon}{\sqrt{p}} \right) T^{-1} \| b \|^2 - T^{-1/2} \| b \| \quad O_P \left( 1 \right) > 0 \]

if \( T^{-1/2} \| b \| \) is sufficiently large. It follows that \( T^{-1/2} \| \bar{b} \| \) must be stochastically bounded and the result follows if \( T \) is fixed.

In the case of large \( T \), we can show that \( \bar{b}_t = \beta_t^* - \beta_t^0 = O_P \left( N^{-1/2} \right) \) for each \( t \) by the same arguments as used in the second part of the proof of Lemma A.1 as \( \tilde{Q}_{NT} \) is a symmetric block tridiagonal matrix that is asymptotically nonsingular. ■

**Proof of Theorem 4.1.** (i) The proof parallels that of Theorem 3.1 and we only sketch it. Let \( \tilde{b}_t = N^{1/2} (\hat{\beta}_t - \beta_t^0) \) and \( \bar{b} = N^{1/2} (\hat{\beta} - \beta^0) \). Noting that \( \Delta y_{it} - x'_{it} \beta_t + x'_{i,t-1} \beta_{t-1} = \Delta u_{it} - N^{-1/2} \xi_{it} \) where \( \xi_{it} = x'_{it} b_t - x'_{i,t-1} b_{t-1} \), we have

\[
N \left[ V_{2NT, \lambda_2} (\beta) - V_{2NT, \lambda_2} (\beta^0) \right] = \sum_{t=1}^{T} \sum_{i=1}^{N} \xi_{it} x'_{it} \left\{ \frac{1}{N} \sum_{i=1}^{N} z_{it} \xi_{it} \right\} - 2N^{1/2} \sum_{t=1}^{T} \sum_{i=1}^{N} \xi_{it} x'_{it} \left\{ \frac{1}{N} \sum_{i=1}^{N} z_{it} \Delta u_{it} \right\}
\]

\[= b' \tilde{Q}_{NT} b - 2b' \sqrt{N} \tilde{R}_{NT} b \equiv B_1 (b) - 2B_2 (b), \text{ say}.
\]

As in the proof of Theorem 3.1, we can show that \( |T^{-1} B_3 (b)| = O_P \left( N^{1/2} \lambda_2 T^{-1/2} J^{-1/2} \right) T^{-1/2} \| b \| = O_P \left( 1 \right) T^{-1/2} \| b \| \) and w.p.a.1

\[ |B_1 (b) - 2B_2 (b) + B_3 (b)| / T \geq \lambda_{\min} \left( \tilde{Q}_{NT} \right) T^{-1} \| b \|^2 - O_P \left( 1 \right) T^{-1/2} \| b \| > 0 \]

if \( T^{-1/2} \| b \| = L \) is sufficiently large. Consequently, \( N \left[ V_{2NT, \lambda_2} (\beta) - V_{2NT, \lambda_2} (\beta^0) \right] > 0 \) w.p.a.1 for large \( L \) and \( V_{2NT, \lambda_2} (\beta) \) cannot be minimized in this case. This further implies that \( T^{-1/2} \| \bar{b} \| \) has to be stochastically bounded.
(ii) The result follows from (i) in the case of fixed $T$. In the case of large $T$, the proof is analogous to that of the second part of Theorem 3.1 by utilizing the fact that $\tilde{Q}_{Nt}$ is an asymptotically nonsingular symmetric block tridiagonal matrix. ■

**Proof of Theorem 4.2.** We want to demonstrate that

$$P \left( \left\| \hat{\theta}_t \right\| = 0 \text{ for all } t \in T^0_{m^0} \right) \to 1 \text{ as } N \to \infty. \quad (B.1)$$

Suppose that to the contrary, $\hat{\theta}_t = \hat{\beta}_t - \tilde{\beta}_{t-1} \neq 0$ for some $t \in T^0_{m^0}$ for sufficiently large $N$ or $(N, T)$. To consider the optimization conditions wrt $\beta_t$, $t \geq 2$, based on subdifferential calculus (e.g., Bersekas (1995, Appendix B.5)), we distinguish two cases: (a) $2 \leq t \leq T - 1$ and (b) $t = T$ and $T \in T^0_{m^0}$.

In case (a), we consider two subcases: (a1) $t + 1 = T^0_j \in T^0_{m^0}$ for some $j = 1, \ldots, m^0$, and (a2) $t + 1 \in T^0_{m^0}$.

In either case, we can apply the FOC wrt $\beta_t$ and the equality $\Delta y_{it} = \beta_t' x_{it} - \beta_{t-1}' x_{i,t-1} + \Delta u_{it}$ to obtain

$$0_{p \times 1} = -2 \sum_{i=1}^{N} x_{it} \hat{\theta}_t W_t + \frac{1}{\sqrt{N}} \sum_{i=1}^{N} z_{it} \left[ \Delta y_{it} - \hat{\beta}_t' x_{it} + \hat{\beta}_t' x_{i,t-1} \right]$$

$$+ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_{it} \hat{\theta}_t W_t + \frac{1}{\sqrt{N}} \sum_{i=1}^{N} z_{i,t+1} \left[ \Delta y_{i,t+1} - \hat{\beta}_{t+1}' x_{i,t+1} + \hat{\beta}_{t+1}' x_{i,t} \right]$$

$$+ \sqrt{N} \lambda_1 \hat{\theta}_t - \sqrt{N} \lambda_1 \hat{\theta}_t e_{t+1}$$

$$= -2 \phi_{x,t} W_t \sum_{i=1}^{N} z_{i,t} \left[ \Delta u_{it} - \left( \hat{\beta}_t - \beta_0 \right)' x_{it} + \left( \hat{\beta}_{t-1} - \beta_0 \right)' x_{i,t-1} \right]$$

$$+ 2 \phi_{x,t} W_t \sum_{i=1}^{N} z_{i,t+1} \left[ \Delta u_{i,t+1} - \left( \hat{\beta}_{t+1} - \beta_0 \right)' x_{i,t+1} + \left( \hat{\beta}_t - \beta_0 \right)' x_{it} \right]$$

$$+ \sqrt{N} \lambda_2 \hat{\theta}_t - \sqrt{N} \lambda_2 \hat{\theta}_t e_{t+1}$$

$$= -2 \sqrt{N} \phi_{x,t+1} W_t + \phi_{x,t+1} \left( \hat{\beta}_{t+1} - \beta_0 \right)' W_t + \phi_{x,t+1} \left( \hat{\beta}_t - \beta_0 \right)'$$

$$- \phi_{x,t+1} W_t + \phi_{x,t+1} \left( \hat{\beta}_{t+1} - \beta_0 \right)' + \phi_{x,t+1} W_t + \phi_{x,t+1} \left( \hat{\beta}_t - \beta_0 \right)'$$

$$+ 2 \sqrt{N} \phi_{x,t+1} W_t + \phi_{x,t+1} \left( \hat{\beta}_{t+1} - \beta_0 \right)' + \phi_{x,t+1} W_t + \phi_{x,t+1} \left( \hat{\beta}_t - \beta_0 \right)'$$

$$= B_{tt} + B_{tt} + B_{tt} - B_{tt}, \text{ say},$$

where $\hat{\theta}_{t+1} = \hat{\theta}_{t+1}' \left\| \hat{\theta}_{t+1} \right\|$ if $\left\| \hat{\theta}_{t+1} \right\| \neq 0$ and $\left\| \hat{\theta}_{t+1} \right\| \leq 1$ otherwise.

Since $\hat{\theta}_t \neq 0$, there exists $r \in \{1, \ldots, p\}$ such that $\left\| \hat{\theta}_t \right\| = \max \left\{ \left\| \hat{\theta}_l \right\|, l = 1, \ldots, p \right\}$, where for any $p \times 1$ vector $a_t$, $a_{t,l}$ denotes its $l$th element. Wlog assume that $r = p$, implying that $\left\| \hat{\theta}_t \right\| / \left\| \hat{\theta}_t \right\| \geq 1 / \sqrt{p}$. By Assumptions B.1(i)-(ii) and Theorem 4.1, $B_{tt} = O_P(1)$ and $B_{tt} = O_P(1)$. In view of the fact that $\hat{\theta}_t^{-1} = O_P(N^{-\alpha_2/2})$ for $t \in T^0_{m^0}$, $B_{tt} \geq \sqrt{N} \lambda_2 \hat{\theta}_t / \sqrt{p}$, which is explosive in probability under Assumption B.2(iii) $N^{(\alpha_2+1)/2} \lambda_2 \to \infty$. To bound the probability order of $B_{tt}$, we distinguish
two subcases. In subcase (a1), noting that $\beta_{t+1} - \beta_t \xrightarrow{P} \theta^\prime_{t+1} \neq 0$ by Theorem 4.1, we have $\tilde{w}_{t+1} = \|\theta^\prime_{t+1} + O_P(N^{-1/2})\|^{-n_2} = O_P(J^{-n_2}_{\min})$ and $B_{it} = \sqrt{N} \lambda_2 \tilde{w}_{t+1} \tilde{e}_{t+1,p} = O_P(\sqrt{N} \lambda_2 J^{-n_2}_{\min}) = O_P(1)$. Consequently, $|B_{it,p}| \gg |B_{it,p} + B_{it,p} + B_{it,p}|$ so that (B.2) cannot be true for sufficiently large $N$ or $(N,T)$. Then we conclude that w.p.a.1, $\hat{\theta}_t$ must be in a position where $\|\hat{\theta}_t\|$ is not differentiable in subcase (a1). In addition, a direct application of this result is that if $T^0_j - \tilde{1} \in T^0_m$ for some $j = 1, ..., m^0$, then $P\left(\|\tilde{\theta}_{T^0_j-1}\| = 0\right) \rightarrow 1$ as $N \rightarrow \infty$ and $\sqrt{N} \lambda_2 \tilde{w}_{T^0_j-1} \tilde{e}_{T^0_j-1} = O_P(1)$ in order for the FOC to hold for $t = T^0_j - 1$.

In subcase (a2), we apply deductive arguments as used in the proof of Theorem 3.2 and the result in subcase (a1) so show that $\hat{\theta}_t$ must be in a position that $\|\hat{\theta}_t\|$ is not differentiable for all $t \in T^0_m$ and $t \neq T$.

In case (b), noting that only one term in the penalty term $(\lambda_2 \sum_{t=2}^{T} \tilde{w}_t \|\beta_t - \beta_{t-1}\|)$ is involved with $\beta_T$, it is easy to show that $\theta_T = \beta_T - \beta_{T-1}$ must be in a position where $\|\hat{\theta}_T\|$ is not differentiable if $T \in T^0_m$. Consequently (B.1) follows. ■

Proof of Corollary 4.3. The proof is analogous to that of Corollary 3.3 by using Theorems 4.1-4.2 instead. ■

Proof of Theorem 4.4. Note that $\hat{\alpha}^p_m(\tilde{T}_m) = (\hat{\alpha}_1^p(\tilde{T}_m)', ..., \hat{\alpha}_m^p(\tilde{T}_m)')' = \arg \min_{\alpha_m} Q_{2NT}(\alpha_m; \tilde{T}_m)$. The first order conditions for this minimization problem are

$$0_{p \times 1} = -\frac{2}{N} \sum_{t=2}^{T} \sum_{i=1}^{N} \Delta x_{it} z_{it} W_{1}^{p} \frac{1}{N} \sum_{t=2}^{T} \sum_{i=1}^{N} z_{it} \left(\Delta y_{it} - \hat{\alpha}_1^p \Delta x_{it}\right)$$

$$+ \frac{2}{N} \sum_{i=1}^{N} x_{i,1-1}^t \tilde{z}_{i,1-1} \tilde{W}_{1}^{p} \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{i=1}^{N} z_{it} \left(\Delta y_{it} - \hat{\alpha}_1^p \tilde{x}_{i,1-1}\tilde{t}\right),$$

$$0_{p \times 1} = -\frac{2}{N} \sum_{t=2}^{T} \sum_{i=1}^{N} \Delta x_{it} z_{it} W_{1}^{p} \frac{1}{N} \sum_{t=2}^{T} \sum_{i=1}^{N} z_{it} \left(\Delta y_{it} - \hat{\alpha}_j^p \Delta x_{it}\right)$$

$$+ \frac{2}{N} \sum_{i=1}^{N} x_{i,j-1}^t \tilde{z}_{i,j-1} \tilde{W}_{j-1} \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{i=1}^{N} z_{it} \left(\Delta y_{it} - \hat{\alpha}_j^p \tilde{x}_{i,j-1}\tilde{t}\right);$$

$$0_{p \times 1} = -\frac{2}{N} \sum_{t=2}^{T} \sum_{i=1}^{N} \Delta x_{it} z_{it} W_{m+1}^{p} \frac{1}{N} \sum_{t=2}^{T} \sum_{i=1}^{N} z_{it} \left(\Delta y_{it} - \hat{\alpha}_m^p \Delta x_{it}\right)$$

$$- \frac{2}{N} \sum_{i=1}^{N} x_{i,m} z_{i,m} W_{m+1} \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{i=1}^{N} z_{it} \left(\Delta y_{it} - \hat{\alpha}_m^p \tilde{x}_{i,m}\tilde{t}\right),$$

where we suppress the dependence of $\hat{\alpha}_j^p$'s on $\tilde{T}_m$. 42
Let \( \hat{\phi}_{ab,l} = \frac{1}{N} \sum_{t=1}^{T_l-1} \sum_{j=1}^{N} a_{ij} b_{ij}' \) for \( l = 1, ..., \hat{m} + 1 \), and \( a, b = \Delta x, x, \text{ or } \Delta y \). Let \( \phi_{ab,l+1} = \phi_{ab,l} W_{T_l} \phi_{ab,l+1} \) for \( l = 1, \ldots, \hat{m} \). One can readily solve for \( \hat{\alpha}_m^p \) to obtain \( \hat{\alpha}_m^p = \hat{\gamma}^{-1}_m \hat{\zeta}_y^u \), where

\[
\hat{\gamma}_m = \text{Tr}(D \hat{\gamma}_m) = \left( \hat{\gamma}_{y,1}, \hat{\gamma}_{y,2}, \ldots, \hat{\gamma}_{y,\hat{m}+1} \right)'
\]

By Corollary 4.3, \( \hat{\alpha}_m^p (\hat{\gamma}_m) = \hat{\alpha}_m^p (\gamma_m) \) w.p.a.1. Therefore we can study the asymptotic distribution of \( \hat{\alpha}_m^p (\hat{\gamma}_m) \) by studying that of \( \hat{\alpha}_m^p (\gamma_m) \). Note that \( \hat{\alpha}_m^p (\gamma_m) = \gamma_1^{-1} \Xi_1 \), where \( \gamma_1 \) and \( \Xi_1 \) are defined in (4.1). It is easy to verify that

\[
\sqrt{N} \Xi_1 = \left( \frac{1}{\gamma_1} \right)^{-1} \sqrt{N} D_{m+1}^{-1} \Xi_1
\]

where \( D_{m+1}^{-1} \) is defined in (4.1). Then by Assumption B.3(i), \( D_{m+1}^{-1} \gamma_1 \Xi_1 \Xi_1 D_{m+1}^{-1} \rightarrow \gamma_1 \Xi_1 \). By Assumption B.3(ii), \( \sqrt{N} D_{m+1}^{-1} \Xi_1 \rightarrow D \) N(0, \( \Sigma_0 \)). Then by the Slutsky lemma,

\[
\sqrt{N} D_{m+1}^{-1} (\hat{\alpha}_m^p (\gamma_m) - \alpha^0) \rightarrow D \) N(0, \( \gamma_1 \Xi_1 \)).

This completes the proof of the theorem. ■

**Proof of Theorem 4.5.** The proof is analogous to that of Theorem 3.5 and thus omitted. ■

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